COMPLEX MATRICES AND SPATIAL ROTATIONS

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Contents

1. Introduction and Definitions

This article explores the connection between 2×2 complex matrices and rotations of ordinary 3D space. Specifically, I will show that SU(2), the group of 2×2 special unitary matrices,[1] can act on the sphere via rotations, with each possible rotation corresponding to two SU(2) matrices. The group of rotations of the sphere (or equivalently, of 3D space) is denoted SO(3), so another way of stating the above is that there is a surjective two-to-one homomorphism from SU(2) to SO(3).

This two-to-one correspondence is very important in quantum mechanics, where certain state vectors (known as "spinors") transform under SU(2) matrices, while other state vectors (known as "vectors") transform under the corresponding SO(3) matrices. For example, the spin state of a fermion (a class of particles that includes electrons) is modeled as a spinor, while the spin state of a boson (a class of particles that includes photons) is modeled as a vector.

Although it is possible to demonstrate the correspondence between SU(2) matrices and rotations in a number of ways (quaternions, Clifford algebras, Pauli vectors, homogenous polynomials, and so on[2]), the proof in this article makes use of the SU(2) matrices' identity as complex matrices, which have a natural action on complex projective space. By defining a bijection between the complex projective line and the unit sphere, we obtain an action of SU(2) (and in fact, all of $\mathrm{GL}(2, \mathbb{C})$) on the sphere. I do not assume much prior knowledge on the part of reader (only the basics of linear algebra and the complex numbers), so this article also serves as an introduction to (one-dimensional) projective space, Möbius transformations, and the Riemann sphere.

Before getting into the main proof, I will go over some preliminary definitions.

[1] If you don't know what "special unitary" means, see Definition 1.2 below.

[2] For an elementary exposition of the first three approaches, see [1]. For the homogenous polynomial approach, see [2, Example 4.10] with $m = 2$.

Definition 1.1. The **conjugate transpose** of a complex matrix A is the matrix $A^\dagger$ constructed by transposing A and then complex-conjugating each entry. For example,

$$\begin{pmatrix} 1 & 3i \\ 1+i & 2-i \end{pmatrix}^\dagger = \begin{pmatrix} 1 & 1-i \\ -3i & 2+i \end{pmatrix}.$$

Remark. Note that $(A^\dagger)^\dagger = A$ for all matrices A. Note also that $(AB)^\dagger = B^\dagger A^\dagger$; this property carries over from the corresponding property of the transpose.

Definition 1.2. A square complex matrix A is **unitary** if $AA^\dagger = I$, where I is the identity matrix, and **special unitary** if it also has determinant 1. The set of $n \times n$ special unitary matrices is denoted $\mathrm{SU}(n)$.

Lemma 1.3. $\mathrm{SU}(n)$ *is a group. That is,* $\mathrm{SU}(n)$ *includes the identity matrix, and is closed under composition and inversion.*

Proof. I is special unitary because $\det(I) = 1$ and $II^\dagger = II = I$. Furthermore, if A and B are both special unitary, then $\det(AB) = \det(A)\det(B) = 1 \cdot 1 = 1$, and

$$AB(AB)^\dagger = ABB^\dagger A^\dagger = AIA^\dagger = AA^\dagger = I,$$

so AB is special unitary. Finally, if A is special unitary, then $\det(A^{-1}) = \det(A)^{-1} = 1^{-1} = 1$, and

$$A^{-1}(A^{-1})^\dagger = A^{-1}(A^\dagger)^\dagger = A^{-1}A = I,$$

so A^{-1} is special unitary. □

In the following two sections, I will define bijections between various spaces, which will then give rise to the connection between SU(2) and SO(3). The primary purpose of Section 2 is to give the reader some intuition for the concepts in Section 3, which deal with the complex numbers rather than the real numbers.

2. Real Projective Space

Definition 2.1. The **real projective line** $\mathbb{RP}^1$ is the set of lines through the origin in $\mathbb{R}^2$. The line passing through a point $(x, y) \neq (0, 0)$ is denoted $[x : y]$. Note that $[x : y] = [kx : ky]$ for all $k \neq 0$, since the points (x, y) and (kx, ky) fall on the same line.

Lines through the origin in $\mathbb{R}^2$ (i.e. elements of $\mathbb{RP}^1$) are identified uniquely by their slope, which is an element of $\mathbb{R} \cup \{\infty\}$. The slope of the line $[x : y]$ is y/x, or ∞ if $x = 0$. (Note that $ky/kx = y/x$, so the slope is well-defined.) This constitutes a bijection between $\mathbb{RP}^1$ and $\mathbb{R} \cup \{\infty\}$, the latter of which I will henceforth denote $\hat{\mathbb{R}}$.

$\hat{\mathbb{R}}$ is equivalent to the unit circle via stereographic projection, as shown in Figure 1. Given a point $x \in \mathbb{R}$, we draw a line in $\mathbb{R}^2$ connecting $(x, 0)$ and $(0, 1)$. This line intersects the unit circle at exactly one point, the coordinates of which can be calculated via elementary algebra:

$$\left(\frac{2x}{x^2+1}, \frac{x^2-1}{x^2+1} \right).$$

This process creates a bijection from $\mathbb{R}$ to $S^1 - \{(0, 1)\}$, where S^1 denotes the unit circle. To complete the bijection, we can map $\infty \in \hat{\mathbb{R}}$ to the point $(0, 1)$, which

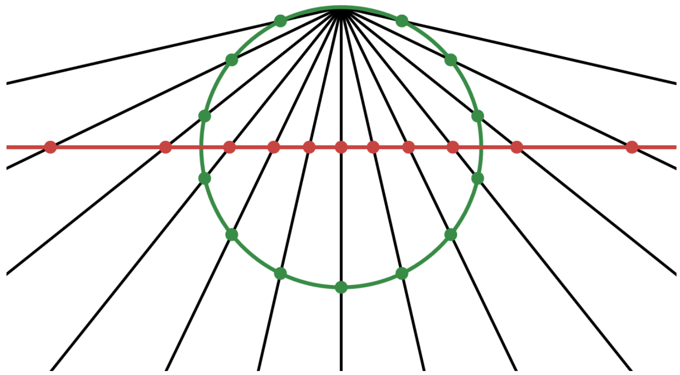

FIGURE 1. The bijection between S^1 (green) and $\hat{\mathbb{R}}$ (red), defined by stereographic projection. The point $(0,1) \in S^1$ corresponds to $\infty \in \hat{\mathbb{R}}$.

is the limit of the above expression as x tends to infinity. The inverse function, which maps S^1 to $\hat{\mathbb{R}}$, is even easier to compute:

$$(x, y) \mapsto \frac{x}{1-y}, \quad (0,1) \mapsto \infty.$$

In summary, we have constructed a chain of bijections $\mathbb{R}\mathrm{P}^1 \simeq \hat{R} \simeq S^1$, the first via taking the slope, and the second via stereographic projection.

3. Complex Projective Space

Now that we have an understanding of $\mathbb{R}\mathrm{P}^1$, it is time to investigate $\mathbb{C}\mathrm{P}^1$, which is defined analogously.

Definition 3.1. The **complex projective line** $\mathbb{C}\mathrm{P}^1$ is the set of complex lines through the origin in $\mathbb{C}^2$. The line passing through a point $(x, y) \neq (0,0)$ is denoted $[x : y]$ and is equal to $\{(kx, ky) : k \in \mathbb{C}\}$. Note that $[x : y] = [kx : ky]$ for all $k \neq 0$.

As in $\mathbb{R}\mathrm{P}^1$, an element $[x : y] \in \mathbb{C}\mathrm{P}^1$ is determined by its slope $y/x \in \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. However, from now on, we will actually use the *coslope* x/y, since that is standard when working with Möbius transformations, which I will introduce in the next section.

Lemma 3.2. *The coslope function $f : \mathbb{C}\mathrm{P}^1 \to \hat{\mathbb{C}}$, $[x : y] \mapsto x/y$ is a bijection. Here we define $x/0$ to be ∞. The inverse of f is given by $g : \hat{\mathbb{C}} \to \mathbb{C}\mathrm{P}^1$, which sends $x \in \mathbb{C}$ to $[x : 1]$ and ∞ to $[1 : 0]$.*

Proof. There are four cases to check:

$$\begin{aligned}
[x : y] \overset{f}{\mapsto} \tfrac{x}{y} \overset{g}{\mapsto} [\tfrac{x}{y} : 1] = [x : y] &\quad \text{if } y \neq 0 \\
[x : 0] \overset{f}{\mapsto} \infty \overset{g}{\mapsto} [1 : 0] = [x : 0] &\quad \text{if } x \neq 0 \\
x \overset{g}{\mapsto} [x : 1] \overset{f}{\mapsto} \tfrac{x}{1} = x &\quad \text{if } x \in \mathbb{C} \\
\infty \overset{g}{\mapsto} [1 : 0] \overset{f}{\mapsto} \tfrac{1}{0} = \infty. &
\end{aligned}$$

□

Just as $\hat{\mathbb{R}}$ is equivalent to the unit circle, $\hat{\mathbb{C}}$ is equivalent to the unit sphere, which we denote S^2. The maps between $\hat{\mathbb{C}}$ and S^2 can be derived from the maps between $\hat{\mathbb{R}}$ and S^1 given in the last section. Explicitly, $\varphi : \hat{\mathbb{C}} \to S^2$ is given by

$$x + iy \mapsto \left(\frac{2x}{r^2+1}, \frac{2y}{r^2+1}, \frac{r^2-1}{r^2+1} \right), \quad \infty \mapsto (0,0,1)$$

and $\varphi^{-1} : S^2 \to \hat{\mathbb{C}}$, which geometrically is stereographic projection, is given by

$$(x, y, z) \mapsto \frac{x+iy}{1-z}, \quad (0,0,1) \mapsto \infty.$$

In the formula for φ, r is defined as $|x + iy|$, so $r^2 = x^2 + y^2$. Note that φ does in fact map into the unit sphere, since

$$\frac{(2x)^2 + (2y)^2 + (r^2-1)^2}{(r^2+1)^2} = \frac{4r^2 + r^4 - 2r^2 + 1}{r^4 + 2r^2 + 1} = 1.$$

One can mechanically verify that φ and φ^{-1} are in fact inverses, but this is rather tedious, so I will omit it.

4. Möbius Transformations

Invertible matrices map lines to lines, so there is a natural action of $\mathrm{GL}(2, \mathbb{C})$ — the group of invertible 2×2 complex matrices — on $\mathbb{C}\mathrm{P}^1$. Explicitly, this action is given by the following:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot [x : y] = [ax + by : cx + dy].$$

In the last section we constructed bijections $\mathbb{C}\mathrm{P}^1 \simeq \hat{\mathbb{C}}$ and $\hat{\mathbb{C}} \simeq S^2$. We can use these bijections to convert the action of $\mathrm{GL}(2, \mathbb{C})$ on $\mathbb{C}\mathrm{P}^1$ into actions on $\hat{\mathbb{C}}$ and S^2. The action on $\hat{\mathbb{C}}$ is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \frac{x}{y} = \frac{ax+by}{cx+dy} = \frac{a\frac{x}{y}+b}{c\frac{x}{y}+d},$$

or in other words

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}.$$

A transformation of this type — that is, a quotient of two linear polynomials — is called a **Möbius transformation**.

Unfortunately, the action of $\mathrm{GL}(2, \mathbb{C})$ on S^2 is much more difficult to give an explicit formula for. You could do so, of course, by applying φ^{-1}, followed by a general Möbius transformation, followed by φ, but the resulting formula is rather complicated and not of much use. Instead, I will show that three specific classes of matrices act as rotations of the sphere around the coordinate axes, and then show that these matrices generate $\mathrm{SU}(2)$.

Lemma 4.1. *Matrices of the form*

$$U_z(\theta) = \begin{pmatrix} \exp\left(i\frac{\theta}{2}\right) & 0 \\ 0 & \exp\left(-i\frac{\theta}{2}\right) \end{pmatrix}$$

rotate the sphere by an angle of θ about the z-axis.

Proof. As a Möbius transformation, $U_z(\theta)$ takes $z \in \hat{\mathbb{C}}$ to

$$\frac{\exp\left(i\frac{\theta}{2}\right) z}{\exp\left(-i\frac{\theta}{2}\right)} = \exp(i\theta)z.$$

In other words, $U_z(\theta)$ simply rotates the complex plane by an angle of θ (and leaves ∞ fixed). Therefore, on the sphere, $U_z(\theta)$ acts by applying a stereographic projection, then rotating the resulting plane by θ, and then applying the reverse stereographic projection. Clearly, the result is that the sphere is rotated by θ about the z-axis. □

Lemma 4.2. *Matrices of the form*

$$U_x(\theta) = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & i\sin\left(\frac{\theta}{2}\right) \\ i\sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{pmatrix}$$

rotate the sphere by an angle of θ about the x-axis.

Proof. A rotation by θ about the x-axis is given by the function

$$R(x, y, z) = (x, y\cos\theta - z\sin\theta, y\sin\theta + z\cos\theta).$$

This corresponds to the transformation on $\hat{\mathbb{C}}$ given by $\varphi^{-1} \circ R \circ \varphi$, that is

$$\begin{aligned} x + iy \mapsto \varphi^{-1}&\left(R\left(\frac{2x}{r^2+1}, \frac{2y}{r^2+1}, \frac{r^2-1}{r^2+1}\right)\right) \\ &= \varphi^{-1}\left(\frac{2x}{r^2+1}, \frac{2y\cos\theta - (r^2-1)\sin\theta}{r^2+1}, \frac{2y\sin\theta + (r^2-1)\cos\theta}{r^2+1}\right) \\ &= \frac{2x + i\left(2y\cos\theta - (r^2-1)\sin\theta\right)}{r^2+1}\left(1 - \frac{2y\sin\theta + (r^2-1)\cos\theta}{r^2+1}\right)^{-1} \\ &= \frac{2x + i\left(2y\cos\theta - (r^2-1)\sin\theta\right)}{r^2+1-2y\sin\theta - (r^2-1)\cos\theta} \\ &= \frac{2x + i\left(2y\cos\theta - (r^2-1)\sin\theta\right)}{r^2(1-\cos\theta) + 1 + \cos\theta - 2y\sin\theta}, \end{aligned}$$

where $r^2 = x^2 + y^2$. It remains to show that the expression above is actually the Möbius transformation

$$\frac{\cos\left(\frac{\theta}{2}\right)(x+iy) + i\sin\left(\frac{\theta}{2}\right)}{i\sin\left(\frac{\theta}{2}\right)(x+iy) + \cos\left(\frac{\theta}{2}\right)}.$$

To see this, we perform complex-number division and then apply the double-angle identities:

$$\begin{aligned} \frac{\cos\left(\frac{\theta}{2}\right)(x+iy) + i\sin\left(\frac{\theta}{2}\right)}{i\sin\left(\frac{\theta}{2}\right)(x+iy) + \cos\left(\frac{\theta}{2}\right)} &= \frac{x\cos\left(\frac{\theta}{2}\right) + i\left(y\cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)\right)}{\cos\left(\frac{\theta}{2}\right) - y\sin\left(\frac{\theta}{2}\right) + ix\sin\left(\frac{\theta}{2}\right)} \\ = \frac{x\left(\cos^2\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right)\right) + i\left[y\left(\cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)\right) - (r^2-1)\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\right]}{r^2\sin^2\left(\frac{\theta}{2}\right) + \cos^2\left(\frac{\theta}{2}\right) - 2y\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)}& \\ &= \frac{x + i\left(y\cos\theta - \frac{1}{2}(r^2-1)\sin\theta\right)}{\frac{1}{2}r^2(1-\cos\theta) + \frac{1}{2}(1+\cos\theta) - y\sin\theta} \\ &= \frac{2x + i\left(2y\cos\theta - (r^2-1)\sin\theta\right)}{r^2(1-\cos\theta) + 1 + \cos\theta - 2y\sin\theta}. \end{aligned}$$

□

Lemma 4.3. *Matrices of the form*

$$U_y(\theta) = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & \sin\left(\frac{\theta}{2}\right) \\ -\sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{pmatrix}$$

rotate the sphere by an angle of θ *about the* y*-axis.*

Proof. Note that $U_y(\theta) = U_z\left(\frac{\pi}{2}\right) U_x(\theta)\, U_z\left(\frac{-\pi}{2}\right)$. (This is straightforward to verify.) Therefore, $U_y(\theta)$ acts on the sphere by first rotating it $\pi/2$ radians clockwise about the z-axis, then rotating it θ radians about the x-axis, and then rotating it $\pi/2$ radians counterclockwise about the z-axis. The overall effect is a rotation by θ about the y-axis. □

5. Putting Things Together

The results of the last section imply that the group generated by the three families of matrices $U_x(\theta)$, $U_y(\theta)$, and $U_z(\theta)$ acts on the sphere via rotations. It is easy to check that these matrices are all in SU(2), and one can in fact show that they generate SU(2); see Lemma 5.2 below. Before proving that, however, it is necessary to characterize the elements of SU(2).

Lemma 5.1. *Every special unitary* 2×2 *matrix is of the form*

$$\begin{pmatrix} a & -\overline{c} \\ c & \overline{a} \end{pmatrix}$$

where $a, c \in \mathbb{C}$ *and* $|a|^2 + |c|^2 = 1$.

Proof. Suppose that

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is special unitary; that is, $AA^\dagger = I$ and $\det(A) = 1$. Note that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} \det(A) & 0 \\ 0 & \det(A) \end{pmatrix} = I.$$

Since matrix inverses are unique, this implies that the second matrix above is actually $A^\dagger$; that is,

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{pmatrix},$$

or in other words, $d = \overline{a}$ and $b = -\overline{c}$. The fact that $\det(A) = 1$ then becomes $|a|^2 + |c|^2 = 1$. □

Lemma 5.2. *The three families of matrices* $U_x(\theta)$*,* $U_y(\theta)$*, and* $U_z(\theta)$ *generate* SU(2)*. Therefore,* SU(2) *acts on the sphere via rotations.*

Proof. Let $A \in \mathrm{SU}(2)$ be any special unitary matrix. By Lemma 5.1,

$$A = \begin{pmatrix} a & -\overline{c} \\ c & \overline{a} \end{pmatrix}$$

with $|a|^2 + |c|^2 = 1$. The requirement that $|a|^2 + |c|^2 = 1$ can be restated as "$(|a|, |c|)$ is a point on the unit circle", which implies that $|a| = \cos(\beta)$ and $|c| = \sin(\beta)$ for some angle β. So, writing a and c in polar form, we have

$$a = \cos(\beta)e^{i\alpha} \quad \text{and} \quad c = \sin(\beta)e^{i\gamma}$$

for some α and γ. Now note that

$$U_z(\alpha-\gamma)U_y(-2\beta)U_z(\alpha+\gamma) = \begin{pmatrix} \cos(\beta)e^{i\alpha} & -\sin(\beta)e^{-i\gamma} \\ \sin(\beta)e^{i\gamma} & \cos(\beta)e^{-i\alpha} \end{pmatrix} = A. \qquad \square$$

All that remains is to show that the homomorphism from SU(2) to SO(3) (i.e. the action of SU(2) on the sphere by rotations) is surjective and two-to-one. Surjectivity follows from the fact that rotations about the three coordinate axes generate all of SO(3). This can be proven in much the same way as Lemma 5.2: any rotation can be written as a composition of a z-rotation, followed by a y-rotation, followed by another z-rotation. The proof of two-to-one-ness relies on the nature of projective space:

Lemma 5.3. *For any $A \in \mathrm{SU}(2)$, the set $\{X \in \mathrm{SU}(2) : X$ acts the same as A on the sphere$\}$ is precisely $\{A, -A\}$.*

Proof. Note that A and X act the same on the sphere if and only if $A^{-1}X$ acts as the identity. And $A^{-1}X$ does nothing to the sphere if and only if it does nothing to $\mathbb{C}\mathrm{P}^1$, since the two actions are equivalent by construction. Recalling the definition of $\mathbb{C}\mathrm{P}^1$ as the set of (complex) lines through the origin in $\mathbb{C}^2$, this simply means that $A^{-1}X$ has every vector as an eigenvector, which is only possible if $A^{-1}X = kI$ for some k. Rearranging this equation, we get $X = kA$.

So the question we must answer is: Which matrices $X = kA$ are in SU(2)? Since $A \in \mathrm{SU}(2)$, we know that $\det(A) = 1$, so $\det(X) = \det(kA) = k^2$. For X to be in SU(2), k^2 must be 1, implying that $k = \pm 1$, and thus $X = \pm A$.

Note that $-A$ is in fact unitary, in addition to having determinant 1:

$$-A(-A)^\dagger = -A(-A^\dagger) = AA^\dagger = I. \qquad \square$$

In summary, we have shown the following:

- There is a bijection between $\mathbb{C}\mathrm{P}^1$ and the unit sphere, defined by first taking the coslope and then applying a stereographic projection.
- Under this bijection, the natural action of $\mathrm{GL}(2,\mathbb{C})$ on $\mathbb{C}\mathrm{P}^1$ gives rise to an action on the sphere.
- SU(2) matrices act on the sphere via rotations, and there are exactly two SU(2) matrices for each rotation.

In the process of proving the above, we found explicit formulae for the SU(2) matrices that act via rotation around the coordinate axes.

6. Closing Remarks

Now that I have proven what I set out to prove, I have a few final remarks. (The last few require mathematical knowledge beyond that assumed in the preceding sections to understand.)

- *Real* special unitary matrices are precisely those matrices that act on $\mathbb{R}^n$ via rotations. (Such matrices are also known as *special orthogonal* matrices; this is where the notation SO(3) comes from.) So special unitary matrices are a generalization of rotations to the complex numbers, and in particular, SU(2) can be thought of as the group of rotations of $\mathbb{C}^2$. What we have shown then, is that there are two rotations of $\mathbb{C}^2$ for each rotation of $\mathbb{R}^3$, and in a homomorphic way.

- $U_z(2\pi)$ is not equal to I; it is in fact equal to $-I$. Similarly, $U_x(2\pi)$ and $U_y(2\pi)$ are both equal to $-I$. In other words, the SU(2) matrices that rotate $\mathbb{R}^3$ by 360 degrees (thus returning it to its original position) only rotate $\mathbb{C}^2$ by 180 degrees. In general, an SU(2) matrix that rotates $\mathbb{R}^3$ by θ only rotates $\mathbb{C}^2$ by $\frac{\theta}{2}$. (The formulas for $U_x(\theta)$, $U_y(\theta)$, and $U_z(\theta)$ all contain $\frac{\theta}{2}$ but not θ.)
- In physics, where the spin of an electron is modeled as a "spinor" (a 2D complex vector acted on by SU(2) matrices), this has the effect that, if an electron is rotated 360 degrees, its spin vector is negated, and only returns to its original state under a 720 degree rotation. The value of a spinor thus depends not only only on the final result of the rotation, but also on the path that the rotation took. (See below for a glimpse of how this is formalized in topology.)
- SU(2) has one irreducible representation in each dimension; the homomorphism to SO(3) is simply the three-dimensional representation. The odd-dimensional representations of SU(2) are also representations of SO(3), but the even-dimensional representations are not [2, Sec. 4.7]. The irreducible representation of SU(2) in n dimensions is known as the "spin $\frac{n-1}{2}$ representation". In particular, the standard (2-dimensional) representation of SU(2) is called the spin-$\frac{1}{2}$ representation, because the angles involved are half of what they are in the 3-dimensional representation. This is why electrons and other fermions (particles whose spin is modeled as a spinor) are said to have spin $\frac{1}{2}$.
- It is a theorem in topology that every connected manifold X has a unique simply connected covering manifold $\tilde{X}$. If X is a Lie group, then $\tilde{X}$ can also be given the structure of a Lie group, such that the covering map from $\tilde{X}$ to X is a homomorphism. Concretely, $\tilde{X}$ can be constructed as the space of paths out of the identity in X up to path homotopy, with the covering map $\tilde{X} \to X$ given by taking the endpoint of a path. The connection between SO(3) and SU(2) is a special case of this general result [2, Sec. 5.8].
  - Topologically, SO(3) is real projective 3-space [2, Prop. 1.17], which has fundamental group $\mathbb{Z}/2\mathbb{Z}$.* Meanwhile, SU(2) is the 3-sphere (a consequence of Lemma 5.1), which is simply connected. The latter is the universal cover of the former, and the covering map is two-to-one because $\mathbb{Z}/2\mathbb{Z}$ has two elements.
  - *In particular, the loop in SO(3) corresponding to a 360 degree rotation is *not* homotopic to the identity, while the loop corresponding to a 720 degree rotation is. This can be demonstrated physically with a variety of tricks, such as the plate trick and the belt trick [3].
  - An informal way of stating all of this is that SU(2) is the group of "rotations that keep track of how they got there (up to homotopy)".

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