1 Free extensions

Given a commutative ring R, one can construct the polynomial ring $R[x]$ by freely extending R with an element x. More specifically, $R[x]$ is defined by adding and multiplying the elements of R along with a symbol x in all possible ways, while keeping any preexisting identities among the elements of R and retaining the structure of a commutative ring. This process of "free extension" can apply to other algebraic structures as well. I will write $E(S)$ for the result of freely extending some algebraic structure S. Here are a few examples:

- If R is a commutative ring, $E(R) \cong R[x]$.
- If G is a group, $E(G) \cong G * \mathbb{Z}$, where $*$ denotes the free product.
- If G is an abelian group, $E(G) \cong G \oplus \mathbb{Z}$.
- If R is a ring, $E(R) \cong R * \mathbb{Z}[x]$, where $*$ denotes the free product of rings.

This notation can lead to ambiguity, because the structure of $E(S)$ depends on the structure of S. Therefore, it is often a good idea to explicitly state the type of algebraic structure one is working with. For example, if G is an abelian group, $E_{Ab}(G) \cong G \oplus \mathbb{Z}$, but $E_{Grp}(G) \cong G * \mathbb{Z}$.

2 A categorical perspective

Because the process of free extension applies to any alebraic structure, a natural question to ask is how it can be defined in categorical terms. In other words, given a category C and an object $S \in C$, how can one construct the free extension $E(S) \in C$?

First, let's look at the specific case of commutative rings. For any commutative ring R, here are some of the properties of $R[x]$:

- There is a homomorphism $c_R : R \to R[x]$ that sends each element of R to the corresponding constant polynomial.
- There is an element $x_R \in R[x]$ called the identity polynomial.
- Given a commutative ring Q, an element $q \in Q$, and a homomorphism $f : R \to Q$, there is a unique homomorphism $[q, f] : R[x] \to Q$ such that $[q, f](x_R) = q$ and $[q, f] \circ c_R = f$.

These properties are actually sufficient to define $R[x]$ up to isomorphism. The only thing keeping it from being a *categorical* definition is the mention of elements. But it is fairly simple to translate a discussion about elements into a discussion about morphisms. An element of a ring R is equivalent to a function $1 \to U(R)$, where $U(R)$ is the underlying set. Because the free functor $F : \mathbf{Set} \to \mathbf{CRing}$ is left adjoint to U, a morphism $1 \to U(R)$ is equivalent to a morphism $F(1) \to R$.

By replacing "element of Q" with "morphism $F(1) \to Q$", and writing $E(R)$ instead of $R[x]$, we arrive at the following definition:

- There is a morphism $c_R : R \to E(R)$.
- There is a morphism $x_R : F(1) \to E(R)$.
- Given an object Q, a morphism $q : F(1) \to Q$, and a morphism $f : R \to Q$, there is a unique morphism $[q, f] : E(R) \to Q$ such that $[q, f]x_R = q$ and $[q, f]c_R = f$.

This definition works not only in **CRing**, but in any category C with a free functor $F : \mathbf{Set} \to C$. In fact, there is a much more concise wording of the definition above:

- If it exists, the free extension $E(S)$ of an object $S \in C$ is the coproduct $F(1) + S$.

If $E(S)$ is defined for every object S, then E can be made into a functor, mapping each morphism $f : S \to T$ to a morphism $E(f) : E(S) \to E(T)$, defined as $E(f) = [x_T, c_T f]$. This has the property that $E(f)x_S = x_T$ and $E(f)c_S = c_T f$.

3 Proof: E is a monad

The latter property, namely that $E(f)c_S = c_T f$, implies that there is a natural transformation $c : Id \to E$ with the obvious components.

There is also a natural transformation $\mu : E^2 \to E$ with components $\mu_S = [x_S, id_{E(S)}]$. For each $f : S \to T$, the naturality condition for μ is provable via a long chain of equalities:

$$
\begin{aligned}
\mu_T E^2(f) &= [x_T, id_{E(T)}][x_{E(T)}, c_{E(T)}E(f)] \\
&= [[x_T, id_{E(T)}]x_{E(T)}, [x_T, id_{E(T)}]c_{E(T)}E(f)] \\
&= [x_T, id_{E(T)}E(f)] \\
&= [x_T, E(f)] \\
&= [E(f)x_S, E(f)] \\
&= E(f)[x_S, id_{E(S)}] \\
&= E(f)\mu_S
\end{aligned}
$$

The next step is to show that c and μ satisfy the monad laws.

- Left identity

$$
\begin{aligned}
\mu_S E(c_S) &= [x_S, id_{E(S)}][x_{E(S)}, c_{E(S)}c_S] \\
&= [[x_S, id_{E(S)}]x_{E(S)}, [x_S, id_{E(S)}]c_{E(S)}c_S] \\
&= [x_S, id_{E(S)}c_S] \\
&= [x_S, c_S] \\
&= id_{E(S)}
\end{aligned}
$$

- Right identity

$$
\begin{aligned}
\mu_S c_{E(S)} &= [x_S, id_{E(S)}]c_{E(S)} \\
&= id_{E(S)}
\end{aligned}
$$

- Associativity

$$
\begin{aligned}
\mu_S E(\mu_S) &= [x_S, id_{E(S)}][x_{E(S)}, c_{E(S)}\mu_S] \\
&= [[x_S, id_{E(S)}]x_{E(S)}, [x_S, id_{E(S)}]c_{E(S)}\mu_S] \\
&= [x_S, id_{E(S)}\mu_S] \\
&= [x_S, \mu_S]
\end{aligned}
$$

$$
\begin{aligned}
\mu_S \mu_{E(S)} &= \mu_S[x_{E(S)}, id_{E^2(S)}] \\
&= [\mu_S x_{E(S)}, \mu_S id_{E^2(S)}] \\
&= [[x_S, id_{E(S)}]x_{E(S)}, \mu_S] \\
&= [x_S, \mu_S]
\end{aligned}
$$

and therefore, $\mu_S E(\mu_S) = \mu_S \mu_{E(S)}$.

4 Algebras over E

An algebra over E is an object S along with a morphism $a : E(S) \to S$ such that $ac_S = id_S$ and $aE(a) = a\mu_S$.

Actually, it is only necessary to check the first condition, because if $ac_S = id_S$, then

$$\begin{aligned} aE(a) &= a[x_S, c_S a] \\ &= [ax_S, ac_S a] \\ &= [ax_S, a] \\ &= a[x_S, id_{E(S)}] \\ &= a\mu_S \end{aligned}$$

For each morphism $s : F(1) \to S$, the pairing $[s, id_S] : E(S) \to S$ is an algebra over E. In fact, *every* algebra over E can be formed this way. Any morphism $a : E(S) \to S$ is uniquely defined by the composites ax_S and ac_S (due to the definition of the coproduct), and for a to be an algebra, ac_S must equal the identity. Therefore, for every object S, there is a bijection between $\hom(F(1), S)$ and the set of algebras $E(S) \to S$.

In **CRing**, this implies that for each element r in some commutative ring R, there is a unique homomorphism $R[x] \to R$ that does nothing to constant polynomials and maps x to r. Applying this homomorphism to a polynomial is called *evaluating* that polynomial at r.

Given $s : F(1) \to S$ and $t : F(1) \to T$, the corresponding algebras are $[s, id_S]$ and $[t, id_T]$. A morphism in the Eilenberg-Moore category from $[s, id_S]$ to $[t, id_T]$ is a morphism $f : S \to T$ such that $f[s, id_S] = [t, id_T]E(f)$. This requirement is equivalent to all of the following:

$$\begin{aligned} f[s, id_S] &= [t, id_T]E(f) \\ [fs, f] &= [t, id_T][x_T, c_T f] \\ [fs, f] &= [[t, id_T]x_T, [t, id_T]c_T f] \\ [fs, f] &= [t, f] \\ fs &= t \end{aligned}$$

The last equation means that the morphisms in the Eilenberg-Moore category C^E are also morphisms in the undercategory $F(1)/C$, and vice versa, hence these two categories are isomorphic.

5 More general statements

You may have noticed that the proofs in sections 3 and 4 did not rely on the specific properties of $F(1)$. Therefore, these theorems apply not only to E, but to any coproduct functor. Explicitly, if C is a category with binary coproducts, A is an object in C, and $x_S : A \to A + S$ and $y_S : S \to A + S$ are injections, the functor G defined by $G(S) = A + S$ and $G(f : S \to T) = [x_T, y_T f]$ has the following properties:

- There is a natural transformation $\mu : G^2 \to G$ with components $\mu_S = [x_S, id_{G(S)}]$.
- (G, μ, y) is a monad.
- For every $S \in C$, there is a bijection between $\hom(A, S)$ and the set of algebras $G(S) \to S$. The functions that make up this bijection are $s \mapsto [s, id_S]$ and $a \mapsto a x_S$.
- C^G is isomorphic to A/C.

6 Composition of elements

An important feature of polynomial rings is that one can *compose* their elements; given two polynomials $p, q \in R[x]$, there is a composite $p \circ q$ such that $p(q(r)) = (p \circ q)(r)$ for all $r \in R$. This notion of "composition" generalises to free extensions in any category.

Let C be a category, and $S \in C$ an object with free extension $E(S)$. As shown in section 4, for each $s : F(1) \to S$, there is an algebra $[s, id_S] : E(S) \to S$, which generalises the idea of applying polynomials to constants. By analogy, for each $p : F(1) \to E(S)$, there is a morphism $[p, c_S] : E(S) \to E(S)$, which generalises the idea of applying polynomials to polynomals (hence composing them). I will define the composite of two morphisms $p, q : F(1) \to E(S)$ as $\mathrm{cmp}(p, q) = [q, c_S]p$.

One can prove that cmp is associative and has x_S as an identity.

- Left identity

$$\mathrm{cmp}(x_S, p) = [p, c_S]x_S = p$$

- Right identity

$$\mathrm{cmp}(p, x_S) = [x_S, c_S]p = id_{E(S)}p = p$$

- Associativity

$$\begin{aligned}
\text{cmp}(\text{cmp}(o,p),q) &= [q,c_S]\text{cmp}(o,p) \\
&= [q,c_S][p,c_S]o \\
&= [[q,c_S]p,[q,c_S]c_S]o \\
&= [\text{cmp}(p,q),c_S]o \\
&= \text{cmp}(o,\text{cmp}(p,q))
\end{aligned}$$

Therfore, $(\hom(F(1),E(S)),\text{cmp},x_S)$ is a monoid in **Set**. And because $\hom(F(1),E(S))$ is isomorphic to $U(E(S))$, the latter is also monoid under the same operation.

7 Proof: every $E(f)$ is a homomorphism of monoids

Morphisms of the form $E(f): E(S) \to E(T)$ are monoid homomorphisms with respect to composition. In other words, $E(f)x_S = x_T$ and $E(f)\text{cmp}(p,q) = \text{cmp}(E(f)p, E(f)q)$ for all $p,q : F(1) \to E(S)$.

The first fact is a direct consequence of the definition of E. The second fact can be shown via a chain of equalities:

$$\begin{aligned}
\text{cmp}(E(f)p,E(f)q) &= [E(f)q,c_T]E(f)p \\
&= [E(f)q,c_T][x_T,c_Tf]p \\
&= [[E(f)q,c_T]x_T,[E(f)q,c_T]c_Tf]p \\
&= [E(f)q,c_Tf]p \\
&= [E(f)q,E(f)c_S]p \\
&= E(f)[q,c_S]p \\
&= E(f)\text{cmp}(p,q)
\end{aligned}$$