Deformed G_2 Instantons on Aloff-Wallach Spaces

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July 31, 2026

Abstract

A deformed G_2 instanton, also known as a deformed Donaldson-Thomas connection, is a connection over a 7-dimensional manifold with G_2 structure that satisfies a certain gauge-theoretic equation. An Aloff-Wallach space is a 7-dimensional homogeneous space of the form $\mathrm{SU}(3)/T$ where $T \cong U(1)$. In this thesis, I study deformed G_2 instantons on Aloff-Wallach spaces, making use of a four-dimensional family of G_2 structures which on all but one of the Aloff-Wallach spaces exhausts all SU(3)-invariant coclosed G_2 structures. I also prove some results on the curl operator defined by a G_2 structure, including a Weyl-type asymptotic formula for the number of eigenvalues below a given bound. Lastly, in the appendix, I compute the dimensions of spaces of SU(3)-invariant differential forms and SU(3)-invariant coclosed G_2 structures on the Aloff-Wallach spaces.

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1 Introduction

A deformed G_2 instanton, also known as a deformed Donaldson-Thomas connection, is a connection A over a manifold with G_2 structure φ that satisfies the gauge-theoretic equation $\frac{1}{6}F_A^3 + F_A \wedge \star\varphi = 0$. Deformed G_2 instantons are related to deformed Hermitian Yang-Mills connections over Calabi-Yau manifolds, and play a role in the theory of mirror symmetry for G_2 manifolds. They are mainly studied in the case of complex line bundles or $U(1)$ bundles, but can be defined for higher-rank bundles as well.

An Aloff-Wallach space is a 7-dimensional homogeneous space of the form $\mathrm{SU}(3)/T$ where $T \cong U(1)$. The Aloff-Wallach spaces can be parametrized by pairs of integers $(k, l) \neq (0, 0)$, or (as I do in most of this paper) by rational numbers in the interval $[0, 1]$. Every Aloff-Wallach space admits an at least four dimensional family of $\mathrm{SU}(3)$-invariant coclosed G_2 structures.

The idea of this thesis — investigating deformed G_2 instantons on Aloff-Wallach spaces — was suggested to my advisor Franz Pedit by Jason D. Lotay of the University of Oxford. The two papers most closely related to the present paper are [2], which studies (non-deformed) G_2 instantons on Aloff-Wallach spaces, and [18], which studies deformed G_2 instantons on tri-Sasakian manifolds, including the Aloff-Wallach space X_1.

In Section 2, I begin with an overview of some basic properties of G_2 structures on manifolds. In the subsections 2.1 and 2.2, I study certain differential operators defined by a coclosed G_2 structure φ, among them the "curl" operator curl_φ. I prove some results on the spectrum of curl_φ, including a Weyl-type asymptotic formula for the number of positive or negative eigenvalues below a given magnitude.

In Section 3, I discuss four types of connections, namely G_2 instantons, Hermitian Yang-Mills connections, and the "deformed" variants of these, giving proofs of some of their basic properties as well as the relationships between them.

In Section 4, I describe some prior results on deformed G_2 instantons. These include the alternative G_2 structure $\tilde{\varphi}_A$ associated to a deformed G_2 instanton A (Section 4.1), some aspects of the deformation theory of deformed G_2 instantons (Section 4.2), and a Chern-Simons type functional that has deformed G_2 instantons as critical points (Section 4.3). I show that the Hessian of this functional at a deformed G_2 instanton A is essentially the curl operator of $\tilde{\varphi}_A$, which is never definite.

In Section 5, I show that conjugacy classes of circle subgroups of $\mathrm{SU}(3)$ are in bijection with rational numbers in the interval $[0, 1]$. Then in Section 6, I introduce the Aloff-Wallach spaces, using the notation X_q where $q \in \mathbb{Q} \cap [0, 1]$. Following [2], I define a four-dimensional family of invariant coclosed G_2 structures on X_q, some of which are nearly parallel (in the sense of definition 2.1). For $q \neq 1$, this family exhausts all invariant coclosed G_2 structures on X_q.

In Section 7, I study deformed G_2 instantons on X_q, in accordance with the title of this paper. Specifically, I look at $U(1)$ bundles $P_n \to X_q$ — defined as the tensor of the "tautological" bundle $\mathrm{SU}(3) \to X_q$ with itself n times — along with the aforementioned four-dimensional family of G_2 structures. For $q \neq 1$, my results constitute a complete classification of SU(3)-invariant deformed G_2 instantons on P_n with respect to SU(3)-invariant coclosed G_2 structures. Using the non-isolatedness of some of the deformed G_2 instantons on X_0, we are able to show that a certain cohomological invariant which I denote $\varsigma(X_0, \varphi)$ is nonzero for some of the G_2 structures φ under consideration.

In Section 8, I look at the special case X_1, considering a larger space of connections on P_n (3-dimensional instead of 1-dimensional) in accordance with the larger space of SU(3)-invariant 1-forms on X_1. As the G_2 structure varies, the space of SU(3)-invariant deformed G_2 instantons on P_n undergoes a number of singularities, one of which involves a circle and two points instantaneously becoming a 2-sphere. Deformed G_2 instantons on X_1 were also studied by Lotay and Oliveira [18], but from a tri-Sasakian point of view. In the subsection 8.1, I relate their results to my own.

Lastly, in Appendix A, I compute the dimension of the vector space of SU(3)-invariant p-forms on X_q, for all p and q. I then show that, while X_0 has more SU(3)-invariant G_2 structures than X_q with $0 < q < 1$, it has no more SU(3)-invariant *coclosed* G_2 structures, a fact that was overlooked in [2]. On the other hand, I show that X_1 has a 10-dimensional space of SU(3)-invariant coclosed G_2 structures, more than the other Aloff-Wallach spaces.

2 G_2 Structures on Manifolds

On $\mathbb{R}^7$, we define a 3-form

$$\varphi_0 = e_{123} + e_{145} + e_{167} + e_{246} - e_{257} - e_{347} - e_{356}. \tag{1}$$

(Here $e_1, \dots, e_7$ is the standard basis for the dual space of $\mathbb{R}^7$, and $e_{ijk} = e_i \wedge e_j \wedge e_k$.) The subgroup of $\mathrm{GL}(7, \mathbb{R})$ consisting of linear transformations that preserve φ_0 is known as G_2. This is a 14-dimensional compact Lie group, and is in fact one of the exceptional compact simple Lie groups; its name comes from the fact that it corresponds to the Dynkin diagram conventionally labeled G_2. A fundamental theorem of G_2 geometry states that $G_2 \subseteq \mathrm{SO}(7)$, or in other words, any linear transformation that preserves φ_0 also preserves the standard inner product on $\mathbb{R}^7$ as well as the orientation. G_2 acts transitively on pairs of orthogonal unit vectors in $\mathbb{R}^7$, and also on triples of orthogonal unit vectors (u, v, w) with $\varphi_0(u, v, w) = 0$.

The definition of φ_0 can be motivated by the algebra of the octonions $\mathbb{O}$. On the imaginary octonions (a 7-dimensional vector space denoted $\mathrm{Im}\,\mathbb{O}$), we have a bilinear map $\times : \mathrm{Im}\,\mathbb{O} \times \mathrm{Im}\,\mathbb{O} \to \mathrm{Im}\,\mathbb{O}$ given by $u \times v = \mathrm{Im}(uv) = \frac{1}{2}(uv - vu)$. This shares some properties with the cross product in 3 dimensions (justifying the notation); in particular, $u \times v$ is orthogonal to both u and v, with length equal to the parallelogram spanned by u and v. Thus the multilinear map $(u, v, w) \mapsto \langle u \times v, w \rangle$ is alternating. In fact, under certain bases for $\mathrm{Im}\,\mathbb{O}$, this map is precisely φ_0.

We can also describe φ_0 in terms of the standard SU(3) structure on $\mathbb{C}^3$; I will elaborate on this later.

If V is a 7-dimensional vector space, then a G_2 structure on V is defined as a 3-form $\varphi \in \Lambda^3 V^*$ which is related to φ_0 by a linear isomorphism $\mathbb{R}^7 \to V$. Alternatively, we can define a G_2 structure on V to be a G_2 orbit in the set of bases $\mathbb{R}^7 \to V$. Since $G_2 \subseteq \mathrm{SO}(7)$, a G_2 structure on V determines an inner product as well as an orientation on V.

If M is a 7-dimensional manifold, then a G_2 structure on M is defined as a 3-form $\varphi \in \Omega^3(M)$ which gives a G_2 structure on every tangent space; this is equivalent to a principal G_2 subbundle of the frame bundle of M. A G_2 structure on M determines both a Riemannian metric and an orientation on M. So we can define a 4-form $\psi = \star\varphi \in \Omega^4(M)$; this is sometimes called the "coassociative form" of the G_2 structure, while φ is called the "associative form".

The coassociative form of the standard G_2 structure φ_0 on $\mathbb{R}^7$ (which by definition determines the standard inner product and orientation on $\mathbb{R}^7$) is

$$\psi_0 = e_{4567} + e_{2367} + e_{2345} + e_{1357} - e_{1346} - e_{1256} - e_{1247}. \tag{2}$$

Similar to how a manifold with $U(n)$ structure can be a "Hermitian manifold" or an "almost Kähler manifold" or a "Kähler manifold" depending on the vanishing of various derivatives and torsion tensors, there are several important types of G_2 structure:

Definition 2.1. On a 7-dimensional manifold M, a G_2 structure φ, $\psi = \star\varphi$ is called

- *closed* if $d\varphi = 0$,
- *coclosed* if $d\psi = 0$,
- *parallel* (or *torsion-free*) if $\nabla\varphi = 0$, where ∇ is the Levi-Civita connection,
- *nearly parallel* if $d\varphi = \lambda\psi$ for some $\lambda \in \mathbb{R}^\times$.

If M is a manifold with parallel G_2 structure φ, then the parallel transport around any loop in M preserves φ, so the Riemannian metric on M has holonomy contained in G_2. Along with Spin(7) (realized as a subgroup of SO(8)), G_2 is one of the two exceptional holonomy groups on Berger's list, which means that it is possible for a non-locally-symmetric Riemannian manifold to have holonomy exactly G_2. The first examples of complete Riemannian manifolds with holonomy exactly G_2 were found by Bryant and Salamon [5]; the first compact examples were found by Joyce [12].

In general, a parallel form on a Riemannian manifold is always closed and coclosed, so in a parallel G_2 structure, we have $d\varphi = d\psi = 0$. In fact, the converse also holds [13], so the parallel G_2 structures are precisely those that are both closed and coclosed.

More relevant to the present study are the nearly parallel G_2 structures. Note that a nearly parallel G_2 structure is necessarily not closed (hence not parallel), since we require that $\lambda \neq 0$. On the other hand, every nearly parallel G_2 structure is coclosed, since $d\psi = \frac{1}{\lambda} d^2\varphi = 0$. A nearly parallel G_2 structure on M induces a parallel Spin(7) structure on the Riemannian cone of M, and vice versa, so nearly parallel G_2 structures fall into a paradigm that also includes Sasakian structures (corresponding to Kähler structures on the Riemannian cone) and tri-Sasakian structures (corresponding to hyper-Kähler structures on the Riemannian cone). Since metrics with holonomy contained in Spin(7) are Ricci-flat, one can prove that every nearly parallel G_2 manifold M is positive Einstein, specifically $\mathrm{Ric} = \frac{3}{8}\lambda^2 g$ where Ric is the Ricci tensor and g is the metric [8]. If the metric is complete, this implies that M is compact with trivial first cohomology: $b^1(M) = 0$.

For any G_2 structure φ, ψ on a manifold M, we have the following definition.

Definition 2.2. A submanifold $N \subseteq M$ is called

- *associative* if N is 3-dimensional with φ restricting to the Riemannian volume form on N,
- *coassociative* if N is 4-dimensional with ψ restricting to the Riemannian volume form on N.

The standard associative form φ_0 and coassociative form ψ_0 on $\mathbb{R}^7$ satisfy

$$|\varphi_0(u_1, u_2, u_3)| \leq 1 \qquad |\psi_0(u_1, u_2, u_3, u_4)| \leq 1$$

whenever the $u_i \in \mathbb{R}^7$ are orthogonal unit vectors. So φ, ψ satisfy the same inequalities on every tangent space of M. We conclude that, if the G_2 structure φ is closed, then φ is a calibration, and associative submanifolds are volume-minimizing. Similarly, if the G_2 structure is coclosed, then ψ is a calibration, and coassociative submanifolds are volume-minimizing. (See [17] for an introduction to calibrated geometry.)

Any G_2 structure on a manifold M induces an orthogonal splitting $\Lambda^2 T^*M = \Lambda^2_7 T^*M \oplus \Lambda^2_{14} T^*M$, where the subscripts indicate the ranks of the vector bundles. These can be defined as the eigenspaces of the map $\alpha \mapsto \star(\alpha \wedge \varphi)$, with eigenvalues 2 and -1 respectively.[1] On sections, the decomposition is denoted $\Omega^2(M) = \Omega^2_7(M) \oplus \Omega^2_{14}(M)$. In addition to the eigenspace definition, we have the alternative characterizations

$$\Omega^2_7(M) = \{\star(\alpha \wedge \psi) \mid \alpha \in \Omega^1(M)\}$$
$$\Omega^2_{14}(M) = \{\alpha \in \Omega^2(M) \mid \alpha \wedge \psi = 0\}.$$

Lastly, the decomposition can be understood from a representation theory point of view. The action of G_2 on $\Lambda^2(\mathbb{R}^7)^*$ splits into two irreducible representations: one 7-dimensional and isomorphic to the action of G_2 on $\mathbb{R}^7$, and the other 14-dimensional and isomorphic to the adjoint action of G_2 on its Lie algebra.

[1] Some authors, e.g. [13], use the opposite convention for the orientation determined by a G_2 structure, in which case the eigenvalues are -2 and 1.

A G_2 structure likewise induces an orthogonal splitting of 3-forms into irreducible G_2 representations: $\Lambda^3 T^*M = \Lambda^3_1 T^*M \oplus \Lambda^3_7 T^*M \oplus \Lambda^3_{27} T^*M$. We have the characterizations

$$\begin{aligned}
\Omega^3_1(M) &= \{f\varphi \mid f \in \Omega^0(M)\} \\
\Omega^3_7(M) &= \{\star(\alpha \wedge \varphi) \mid \alpha \in \Omega^1(M)\} \\
\Omega^3_{27}(M) &= \{\alpha \in \Omega^3(M) \mid \alpha \wedge \varphi = \alpha \wedge \psi = 0\}.
\end{aligned}$$

The decompositions of $\Lambda^2 T^*M$ and $\Lambda^3 T^*M$ give decompositions of $\Lambda^5 T^*M$ and $\Lambda^4 T^*M$ by taking the Hodge star. I will use the notation $\pi_k : \Omega^p(M) \to \Omega^p_k(M)$ for the projections.

As hinted at earlier, we can relate the standard G_2 structure φ_0 on $\mathbb{R}^7$ to the standard SU(3) structure on $\mathbb{C}^3$. Specifically, on $\mathbb{C}^3$, we have the complex 1-forms dz_1, dz_2, dz_3, the symplectic form $\omega = \frac{1}{2i}\sum_{j=1}^3 \overline{dz_j} \wedge dz_j$, and the holomorphic volume form $\Upsilon = dz_1 \wedge dz_2 \wedge dz_3$. If we write $dz_j = e_{(2j)} + ie_{(2j+1)}$, then

$$\begin{aligned}
\omega &= e_{23} + e_{45} + e_{67} \\
\tfrac{1}{2}\omega^2 &= e_{4567} + e_{2367} + e_{2345} \\
\mathrm{Re}\Upsilon &= e_{246} - e_{257} - e_{347} - e_{356} \\
\mathrm{Im}\Upsilon &= -e_{357} + e_{346} + e_{256} + e_{247}.
\end{aligned}$$

So, if we identify $\mathbb{R}^7$ with $\mathbb{R} \times \mathbb{C}^3$, we have

$$\begin{aligned}
\varphi_0 &= e_1 \wedge \omega + \mathrm{Re}\Upsilon \\
\psi_0 &= \tfrac{1}{2}\omega^2 - e_1 \wedge \mathrm{Im}\Upsilon.
\end{aligned}$$

On manifolds, this gives the following theorem.

Proposition 2.3. *Let X be a Calabi-Yau 3-fold with Kähler form ω and holomorphic volume form Υ. Let $M \xrightarrow{\pi} X$ be a principal S^1 bundle with connection η, thought of as a real-valued 1-form $\eta \in \Omega^1(M, \mathbb{R})$. Define*

$$\begin{aligned}
\varphi &= \eta \wedge \pi^*\omega + \pi^*\mathrm{Re}\Upsilon \\
\psi &= \tfrac{1}{2}\pi^*\omega^2 - \eta \wedge \pi^*\mathrm{Im}\Upsilon.
\end{aligned}$$

Then φ is a G_2 structure on M with coassociative form ψ.

(Actually, the manifold X in Proposition 2.3 need not be a Calabi-Yau manifold; it need only have an SU(3) structure. However, in the following proposition, we will use the fact that ω and Υ are closed.)

Proposition 2.4. *In the setting of Proposition 2.3, the G_2 structure φ, ψ is*

1. *closed if and only if the connection η is flat,*
2. *coclosed if and only if the curvature $F_\eta \in \Omega^2(X, \mathbb{R})$ has bidegree $(1, 1)$,*

3. never nearly parallel.

(Since the condition for closed is stricter than the condition for coclosed, we see that if the G_2 structure is closed, it is in fact parallel.)

Proof. Note that $d\eta = \pi^* F_\eta$. We compute

$$d\varphi = \pi^*(F_\eta \wedge \omega) \qquad d\psi = -\pi^*(F_\eta \wedge \mathrm{Im}\Upsilon).$$

Since π is a surjective submersion, π^* is injective on forms. Thus $d\varphi = 0$ if and only if $F_\eta \wedge \omega = 0$. Wedging with ω is an isomorphism from $\Omega^2(X)$ to $\Omega^4(X)$ (this is true on any 6-dimensional symplectic manifold), so this means that $F_\eta = 0$.

Similarly, $d\psi = 0$ if and only if $F_\eta \wedge \mathrm{Im}\Upsilon = 0$. Wedging with $\mathrm{Im}\Upsilon$ gives an isomorphism from $\Omega^{2,0}(X)$ to $\Omega^{2,3}(X)$ and from $\Omega^{0,2}(X)$ to $\Omega^{3,2}(X)$, but is 0 on $\Omega^{1,1}(X)$. (This can be seen by noting that on $\mathbb{C}^3$, we have $\mathrm{Im}\Upsilon = \frac{1}{2i}(dz_1 \wedge dz_2 \wedge dz_3 - d\overline{z_1} \wedge d\overline{z_2} \wedge d\overline{z_3})$.) So $F_\eta \wedge \mathrm{Im}\Upsilon = 0$ if and only if $F_\eta \in \Omega^{1,1}(X)$.

Lastly, note that $d\varphi$ cannot be a nonzero multiple of ψ, since ψ has a component that includes η. $\square$

2.1 The Canonical Complex

Let M be a manifold with coclosed G_2 structure φ, ψ. We have a sequence of differential operators

$$\Omega^0(M) \xrightarrow{d_0} \Omega^1(M) \xrightarrow{D_\varphi} \Omega^6(M) \xrightarrow{d_6} \Omega^7(M), \qquad (\#_\varphi)$$

where $D_\varphi(\alpha) = d\alpha \wedge \psi$, and d_0, d_6 are the exterior derivatives indexed by degree. Note that $(\#_\varphi)$ is a chain complex, since ψ is closed. It is in fact an elliptic complex, meaning the sequence of principal symbols is exact away from the zero cross section of T^*M [15]. The adjoints of the operators are

$$d_0^* = -\star d_6 \star \qquad d_6^* = -\star d_0 \star \qquad D_\varphi^* = \star D_\varphi \star. \qquad (3)$$

If we furthermore assume that M is compact, then since $(\#_\varphi)$ is an elliptic complex over a compact manifold, we get Hodge decompositions

$$\Omega^1(M) = \mathrm{im}\, d_0 \oplus \mathrm{im}\, D_\varphi^* \oplus (\ker d_0^* \cap \ker D_\varphi)$$
$$\Omega^6(M) = \mathrm{im}\, d_6^* \oplus \mathrm{im}\, D_\varphi \oplus (\ker d_6 \cap \ker D_\varphi^*),$$

where the third component of each decomposition is finite-dimensional and isomorphic to the respective cohomology. Using the formulas in (3), we see that the decomposition of $\Omega^1(M)$ is related to that of $\Omega^6(M)$ by the Hodge star map; as a result, the two middle cohomologies of $(\#_\varphi)$ have the same dimension.

Recall that we also have a Hodge decomposition for $\Omega^1(M)$ coming from the de Rham complex:

$$\Omega^1(M) = \mathrm{im}\, d_0 \oplus \mathrm{im}\, d_1^* \oplus (\ker d_0^* \cap \ker d_1).$$

Since $\operatorname{im} D_\varphi \subseteq \operatorname{im} d_5$, we know that $\operatorname{im} D_\varphi^* \subseteq \operatorname{im} d_1^*$; we also have $\ker d_1 \subseteq \ker D_\varphi$. So we can refine the two decompositions of $\Omega^1(M)$ to get a fourfold decomposition

$$\Omega^1(M) = \operatorname{im} d_0 \oplus \operatorname{im} D_\varphi^* \oplus (\ker d_0^* \cap \ker d_1) \oplus (\ker D_\varphi \cap \operatorname{im} d_1^*), \tag{4}$$

where the third and fourth components are finite dimensional. I will denote the dimension of the fourth component as $\varsigma(M, \varphi) = \dim(\ker D_\varphi \cap \operatorname{im} d_1^*)$. This can also be expressed as the difference between the dimensions of cohomology groups: $\varsigma(M, \varphi) = \dim H^1(\#_\varphi) - b^1(M)$.

Proposition 2.5. *For a compact manifold M with coclosed G_2 structure φ, the following are equivalent:*

(i) $\varsigma(M, \varphi) = 0$.

(ii) $\ker D_\varphi = \ker d_1$.

(iii) $\operatorname{im} D_\varphi = \operatorname{im} d_5$.

(iv) $\operatorname{im} d_1 \cap \Omega^2_{14}(M) = 0$.

Proof. Note that $\ker d_1$ is the sum of the first and third components in (4), while $\ker D_\varphi$ is the sum of the first, third, and fourth components. So the kernels are equal if and only if the fourth component $\ker D_\varphi \cap \operatorname{im} d_1^*$ is 0, that is $\varsigma(M, \varphi) = 0$. This proves the equivalence of (i) and (ii).

Property (iii) can equivalently be written as $\operatorname{im} D_\varphi^* = \operatorname{im} d_1^*$ by taking the Hodge star. Note that $\operatorname{im} D_\varphi^*$ is the second component in (4), while $\operatorname{im} d_1^*$ is the sum of the second and fourth components. We conclude that (iii) is equivalent to (i).

Lastly, note that (ii) and (iv) can both be stated as: for all $\alpha \in \Omega^1(M)$ such that $d\alpha \wedge \psi = 0$, we have $d\alpha = 0$. $\square$

The properties in Proposition 2.5 are satisfied when the G_2 structure is parallel or nearly parallel:

Proposition 2.6. *If M is a compact manifold with parallel or nearly parallel G_2 structure φ, then $\varsigma(M, \varphi) = 0$.*

Proof. Let ψ be the coassociative form of φ. Since φ is parallel or nearly parallel, we know that $d\varphi = \lambda\psi$ for some (possibly zero) $\lambda \in \mathbb{R}$.

Take some $\alpha \in \Omega^1(M)$ with $d\alpha \wedge \psi = 0$; this implies that $d\alpha \wedge \varphi = -\star d\alpha$ by one of the characterizations of $\Omega^2_{14}(M)$. Thus

$$-d\star d\alpha = d(d\alpha \wedge \varphi) = d\alpha \wedge d\varphi = \lambda d\alpha \wedge \psi = 0.$$

So $\langle\!\langle d\alpha, d\alpha \rangle\!\rangle = \langle\!\langle d^* d\alpha, \alpha \rangle\!\rangle = 0$, which implies that $d\alpha = 0$. In conclusion, $\ker D_\varphi \subseteq \ker d_1$, which means that $\varsigma(M, \varphi) = 0$. $\square$

Although the ς notation is my own, Proposition 2.6 is a known result; see [10, Theorem 9] (noting that the authors' definition of "nearly parallel" includes the parallel case).

2.2 The Curl Operator

On a manifold M with coclosed G_2 structure φ, ψ, we can define an order-1 differential operator $\operatorname{curl}_\varphi : \Omega^1(M) \to \Omega^1(M)$ by

$$\operatorname{curl}_\varphi(\alpha) = \star D_\varphi(\alpha) = \star(d\alpha \wedge \psi).$$

(Compare the curl operator on an oriented Riemannian 3-manifold, which is defined by $\alpha \mapsto \star d\alpha$.) Since the adjoint of D_φ is $\star D_\varphi \star$, we find that $\operatorname{curl}_\varphi$ is self-adjoint. Note that the kernel of $\operatorname{curl}_\varphi$ is the same as the kernel of D_φ, and contains all closed 1-forms (hence is infinite-dimensional). Meanwhile, the image of $\operatorname{curl}_\varphi$ is included in the space of coexact 1-forms $\operatorname{im} d_1^*$, since we are assuming that ψ is closed. Specifically, $\operatorname{curl}_\varphi(\alpha) = d^* \star(\alpha \wedge \psi)$.

From the canonical complex ($\#_\varphi$), we get a self-adjoint elliptic order-2 Laplacian operator $\Delta_\varphi : \Omega^1(M) \to \Omega^1(M)$, which can be written in terms of $\operatorname{curl}_\varphi$:

$$\Delta_\varphi = dd^* + D_\varphi^* D_\varphi = dd^* + \operatorname{curl}_\varphi^2. \tag{5}$$

As an aside, we have the following theorem relating Δ_φ to the Hodge Laplacian $\Delta_d = dd^* + d^*d$.

Proposition 2.7. *If the G_2 structure φ is parallel or nearly parallel, with $d\varphi = \lambda\psi$, then $\Delta_\varphi = \Delta_d + \lambda \operatorname{curl}_\varphi$.*

Proof. In the parallel case, we cite [19, Proposition 4.1.3] to obtain $\Delta_d = dd^* + \operatorname{curl}_\varphi^2$; comparing this with (5), we get $\Delta_\varphi = \Delta_d$. Note that the operators that Suan denotes grad, div, curl, Δ correspond to d, $-d^*$, $\operatorname{curl}_\varphi$, Δ_d via the metric-induced isomorphism between vector fields and 1-forms. (The last of these correspondences relies on the Weitzenböck identity and Ricci-flatness.)

In the nearly parallel case, we cite [8, Corollary 2.8] to obtain $\Delta_d + \lambda \operatorname{curl}_\varphi = dd^* + \operatorname{curl}_\varphi^2$, where the same remarks apply concerning grad, div, curl. Comparing with (5), we get $\Delta_\varphi = \Delta_d + \lambda \operatorname{curl}_\varphi$. □

Inspired by [3], which discusses curl operators $\star d$ on odd-dimensional oriented Riemannian manifolds, I will now prove some properties of the spectrum of $\operatorname{curl}_\varphi$. The following theorem shows that any eigenvector of $\operatorname{curl}_\varphi$ with nonzero eigenvalue is also an eigenvector of Δ_φ.

Proposition 2.8. *Let M be a manifold with coclosed G_2 structure φ. Suppose that $\alpha \in \Omega^1(M)$ is nonzero and satisfies $\operatorname{curl}_\varphi \alpha = \lambda\alpha$ with $\lambda \neq 0$. Then $\Delta_\varphi \alpha = \lambda^2 \alpha$.*

Proof. Since $\alpha = \frac{1}{\lambda} \operatorname{curl}_\varphi \alpha$, we see that $\alpha \in \operatorname{im} d_1^*$, and thus $d^*\alpha = 0$. So by equation (5), we have $\Delta_\varphi \alpha = \operatorname{curl}_\varphi^2 \alpha = \lambda^2 \alpha$. □

Although I am stating things in terms of smooth differential forms, this result should also hold if α belongs to a larger space such as a Sobolev space. Then since Δ_φ is elliptic, we can apply elliptic regularity to find that eigenvectors of $\operatorname{curl}_\varphi$ with nonzero eigenvalue are smooth.

If M is compact, then since Δ_φ is a positive self-adjoint elliptic operator on a compact manifold, its spectrum is an unbounded set of nonnegative real numbers with no accumulation points, and the eigenspace of Δ_φ associated to every eigenvalue is finite-dimensional. Thus we obtain the following theorem.

Proposition 2.9. *On a compact manifold M with coclosed G_2 structure φ, the spectrum of $\operatorname{curl}_\varphi$ is a discrete subset of $\mathbb{R}$ with no accumulation points, and the eigenspace of $\operatorname{curl}_\varphi$ associated to every nonzero eigenvalue is finite dimensional.*

Proof. If λ is a nonzero eigenvalue of $\operatorname{curl}_\varphi$, then by Proposition 2.8, λ^2 is an eigenvalue of Δ_φ. So the spectrum of $\operatorname{curl}_\varphi$ is discrete and has no accumulation points. Moreover, the λ-eigenspace of $\operatorname{curl}_\varphi$ is contained in the λ^2-eigenspace of Δ_φ, hence is finite dimensional. □

An asymptotic formula for the number of positive or negative eigenvalues of $\operatorname{curl}_\varphi$ is given by the following theorem.

Proposition 2.10. *Let M be a compact manifold with coclosed G_2 structure φ. For all $\lambda > 0$, let $N_+(\lambda)$ be the number of positive eigenvalues of $\operatorname{curl}_\varphi$ below λ, and let $N_-(\lambda)$ be the number of negative eigenvalues of $\operatorname{curl}_\varphi$ above $-\lambda$; here we are counting the eigenvalues with multiplicity. Then we have the Weyl asymptotic formulas*

$$N_\pm(\lambda) = \frac{\operatorname{Vol}(M)}{280\pi^4}\lambda^7 + O(\lambda^6).$$

as $\lambda \to \infty$. This implies that the ratio between $N_\pm(\lambda)$ and $\frac{\operatorname{Vol}(M)}{280\pi^4}\lambda^7$ approaches 1 as $\lambda \to \infty$.

Proof. I have based my computation on [3, Theorem 3.6], which cites [11, Theorem 0.1]. Recall that $\Omega^1(M)$ has an orthogonal decomposition as $\ker D_\varphi \oplus \operatorname{im} D_\varphi^*$, which we can also write as $\ker \operatorname{curl}_\varphi \oplus \operatorname{im} \operatorname{curl}_\varphi$. We have the restriction $\operatorname{curl}_\varphi : \operatorname{im} D_\varphi^* \to \operatorname{im} D_\varphi^*$, whose eigenvalues are the nonzero eigenvalues of $\operatorname{curl}_\varphi$. Let $\Pi : \Omega^1(M) \to \Omega^1(M)$ denote the orthogonal projection onto $\operatorname{im} D_\varphi^*$. At $\xi \in T_x^*M$, its principal symbol $\sigma(\Pi)(\xi) : (T_x^*M)_\mathbb{C} \to (T_x^*M)_\mathbb{C}$ is the orthogonal projection onto the image of $\sigma(D_\varphi^*)(\xi) : (\Lambda^6 T_x^*M)_\mathbb{C} \to (T_x^*M)_\mathbb{C}$. By ellipticity of $(\#_\varphi)$, we have

$$\operatorname{im} \sigma(D_\varphi^*)(\xi) = (\operatorname{im} \sigma(d_0)(\xi))^\perp = \xi^\perp,$$

so $\sigma(\Pi)(\xi)$ is the orthogonal projection onto $\xi^\perp$. Meanwhile, $\sigma(\operatorname{curl}_\varphi)(\xi)$ sends $\alpha \in (T_x^*M)_\mathbb{C}$ to $\star(i\xi \wedge \alpha \wedge \psi)$. Using the fact that G_2 acts transitively on unit vectors, we can find an orthonormal basis $e_1, \ldots, e_7$ for T_x^*M with $e_1 = \xi/|\xi|$ in which ψ takes the form (2). Then $\sigma(\operatorname{curl}_\varphi)(\xi)(\alpha) = i|\xi|\star(e_1 \wedge \psi \wedge \alpha)$, and we see that $\sigma(\operatorname{curl}_\varphi)(\xi)$ has the eigenvalues 0 (multiplicity 1, corresponding to the eigenvector ξ), $|\xi|$ (multiplicity 3), and $-|\xi|$ (multiplicity 3).

Let $\hat{\pi}_+(\xi) : (T_x^*M)_\mathbb{C} \to (T_x^*M)_\mathbb{C}$ denote the orthogonal projection onto the eigenspaces of $\sigma(\operatorname{curl}_\varphi)(\xi)$ with eigenvalue in $[0, 1]$; similarly, let $\hat{\pi}_-(\xi)$ denote the projection onto the eigenspaces of $\sigma(\mathrm{curl}_\varphi)(\xi)$ with eigenvalue in $[-1, 0]$. By [11, Theorem 0.1], we have $N_\pm(\lambda) = \kappa_\pm \lambda^7 + O(\lambda^6)$, where

$$\kappa_\pm = (2\pi)^{-7} \int_{T^*M} \mathrm{tr}(\hat{\pi}_\pm(\xi)\sigma(\Pi)(\xi))\, dx d\xi.$$

If $0 < |\xi| \leq 1$, then $\hat{\pi}_\pm(\xi)\sigma(\Pi)(\xi)$ is a projection onto a 3-dimensional space (namely the $\pm|\xi|$ eigenspace of $\sigma(\mathrm{curl}_\varphi)(\xi)$), hence has trace 3. And if $|\xi| > 0$, then $\hat{\pi}_\pm(\xi)\sigma(\Pi)(\xi) = 0$. So we see that

$$\kappa_\pm = (2\pi)^{-7}\,\mathrm{Vol}(M) 3 v_7,$$

where v_7 is the volume of the unit ball in 7 dimensions. This is $v^7 = \frac{16\pi^3}{105}$, so we conclude that

$$\kappa_\pm = \frac{\mathrm{Vol}(M)}{280\pi^4}. \qquad \square$$

As an immediate corollary, we see that the spectrum of curl_φ is unbounded in both directions, and is asymptotically symmetric in the sense that

$$\lim_{\lambda\to\infty} \frac{N_+(\lambda)}{N_-(\lambda)} = 1.$$

Lastly, I will relate curl_φ to another "curl" operator, namely $\star d : \Omega^3(M) \to \Omega^3(M)$, which is studied in [3] (along with its analogues in other dimensions). The author of [3] briefly mentions the operator that I denote curl_φ, but states that "there seems to be no relation" since the vector bundles have different ranks. In fact there is a relation, by way of the rank-7 subbundle $\Omega^3_7(M)$. Specifically, we define $P : \Omega^3_7(M) \to \Omega^3_7(M)$ by $P(\alpha) = \pi_7(\star d\alpha)$, and recall that there is an isomorphism $f : \Omega^1(M) \to \Omega^3_7(M)$ given by $f(\alpha) = \star(\alpha \wedge \varphi)$. Then in the (nearly) parallel case, we have the following theorem relating P to curl_φ.

Proposition 2.11. *Let M be a manifold with parallel or nearly parallel G_2 structure, so that $d\varphi = \lambda\psi$ for some $\lambda \in \mathbb{R}$. Then for all $\alpha \in \Omega^1(M)$, we have*

$$f^{-1}(P(f(\alpha))) = \frac{1}{2}\,\mathrm{curl}_\varphi(\alpha) + \frac{\lambda}{4}\alpha.$$

where P, f are defined as above. (In the parallel case, $\lambda = 0$, so $\mathrm{curl}_\varphi = f^{-1} \circ 2P \circ f$.)

Proof. In the nearly parallel case, we cite [8, Lemma 2.11] and find that, for all $\alpha \in \Omega^1(M)$,

$$-\pi_7 d(\alpha^\sharp \lrcorner \psi) = \left(\frac{1}{2}\,\mathrm{curl}_\varphi(\alpha) + \frac{\lambda}{4}\alpha\right) \wedge \varphi,$$

where we are using the decomposition of $\Omega^4(M)$. Taking the Hodge star of both sides and using the fact that $\alpha^\sharp \lrcorner \psi = -\star(\alpha \wedge \varphi) = -f(\alpha)$, we get

$$P(f(\alpha)) = f\left(\frac{1}{2}\,\mathrm{curl}_\varphi(\alpha) + \frac{\lambda}{4}\alpha\right).$$

In the parallel case, we cite [7, Proposition 2.25], specifically the section of the proof listed under (iii), $k = 3$. One must account for the fact that the the cited paper uses the opposite orientation convention, which means that $\star$ and ψ are negated (while curl_φ is unchanged). $\square$

The curl operator and its spectrum has some relevance to the theory of deformed G_2 instantons, since it appears in the Hessian of a functional which has deformed G_2 instantons as its critical points; see Section 4.3.

3 Connections on Manifolds with G_2 Structure

3.1 The G_2 Instanton Equation

Definition 3.1 (G_2 instanton)**.** Let (M, φ, ψ) be a manifold with G_2 structure, and let $P \to M$ be a principal G-bundle, for some Lie group G. A connection A on P is called a G_2 instanton if its curvature form $F_A \in \Omega^2(M, \mathrm{ad}(P))$ is in fact in $\Omega^2_{14}(M, \mathrm{ad}(P))$. This means that $F_A \wedge \psi = 0$, or equivalently, $\star F_A = -F_A \wedge \varphi$.

If the G_2 structure is closed or nearly parallel, then every G_2 instanton A is Yang-Mills in the sense that $d_A(\star F_A) = 0$. Indeed, closed and nearly parallel G_2 structures both satisfy the equation $d\varphi = \lambda\psi$ for some $\lambda \in \mathbb{R}$. If this is the case and A is a G_2 instanton, then

$$d_A(\star F_A) = d_A(-F_A \wedge \varphi) = -(d_A F_A) \wedge \varphi - F_A \wedge (d\varphi) = -\lambda F_A \wedge \psi = 0.$$

Moreover, in the closed case, G_2 instantons have the stronger property that they are energy-minimizing:

Proposition 3.2. *Let (M, φ) be a compact manifold with closed G_2 structure. Let G be a Lie group equipped with a G-invariant inner product on its Lie algebra* $\mathrm{Lie}(G)$*, and let $P \to M$ be a principal G-bundle. Given a connection A on P, we can define its energy as $\|F_A\|^2 = \int_M \langle F_A, F_A\rangle \mathrm{vol}_M$, where the inner product comes from the Riemannian metric on M as well as the inner product on* $\mathrm{Lie}(G)$*. Then G_2 instantons, if they exist, have the least energy among all connections on P.*

Proof. Let A be a connection on P. By Chern-Weil theory, the cohomology class of $\langle F_A \wedge F_A\rangle \in \Omega^4(M)$ does not depend on A. Then since φ is closed, the integral $\int_M \langle F_A \wedge F_A\rangle \wedge \varphi$ does not depend on A; call this $c(P)$. Write $F_A = \pi_7 F_A + \pi_{14} F_A$, where $\star(\pi_7 F_A \wedge \varphi) = 2\pi_7 F_A$ and $\star(\pi_{14} F_A \wedge \varphi) = -\pi_{14} F_A$.

Then $\|F_A\|^2 = \|\pi_7 F_A\|^2 + \|\pi_{14} F_A\|^2$, and

$$\begin{aligned}
c(P) &= \int_M \langle \pi_7 F_A \wedge \pi_7 F_A \rangle \wedge \varphi + \int_M \langle \pi_{14} F_A \wedge \pi_{14} F_A \rangle \wedge \varphi \\
&= \int_M \langle \pi_7 F_A \wedge (\pi_7 F_A \wedge \varphi) \rangle + \int_M \langle \pi_{14} F_A \wedge (\pi_{14} F_A \wedge \varphi) \rangle \\
&= 2\int_M \langle \pi_7 F_A \wedge \star \pi_7 F_A \rangle - \int_M \langle \pi_{14} F_A \wedge \star \pi_{14} F_A \rangle \\
&= 2\int_M \langle \pi_7 F_A, \pi_7 F_A \rangle \mathrm{vol}_M - \int_M \langle \pi_{14} F_A, \pi_{14} F_A \rangle \mathrm{vol}_M \\
&= 2\|\pi_7 F_A\|^2 - \|\pi_{14} F_A\|^2.
\end{aligned}$$

(The exclusion of the "cross terms" in the first line is justified by the fourth line: if we had included the cross terms, we would find that they are equal to integrals of $\langle \pi_7 F_A, \pi_{14} F_A \rangle = 0$.) We can combine these equations to get

$$\|F_A\|^2 = 3\|\pi_7 F_A\|^2 - c(P).$$

So G_2 instantons (connections A satisfying $\|\pi_7 F_A\|^2 = 0$) have the least energy among all connections. Moreover, they have energy $-c(P)$, and can only exist on bundles with $c(P) \leq 0$. □

G_2 instantons over nearly parallel G_2 manifolds, although they are Yang-Mills, need not be energy minimizing. Indeed, in [2], the authors find G_2 instantons over the Aloff-Wallach space X_0 equipped with a nearly parallel G_2 structure which are not even locally energy-minimizing, instead being saddle points of the energy functional.

G_2 instantons are related to another type of connection, namely Hermitian Yang-Mills connections. These are defined as follows.

Definition 3.3 (Hermitian Yang-Mills connection)**.** Let M be an n-complex-dimensional Kähler manifold with Kähler form ω. Let $E \to M$ be a rank-r Hermitian vector bundle over M. Note that E corresponds to a principal $U(r)$ bundle $P \to M$ (namely its unitary frame bundle), and $\mathrm{ad}(P)$ is canonically isomorphic to the subbundle $\mathfrak{u}(E) \subseteq \mathrm{End}_{\mathbb{C}}(E)$ consisting of anti-self-adjoint maps. A unitary connection A on E, which is the same as a connection A on P, has curvature form $F_A \in \Omega^2(M, \mathfrak{u}(E))$. We call A a Hermitian Yang-Mills connection if F_A has bidegree $(1,1)$ and

$$F_A \wedge \star\omega = \lambda \mathrm{vol}_M \otimes iI. \tag{6}$$

for some constant $\lambda \in \mathbb{R}$. Since $\star\omega = \frac{\omega^{n-1}}{(n-1)!}$ and $\mathrm{vol}_M = \frac{\omega^n}{n!}$, we can also write equation (6) as

$$nF_A \wedge \omega^{n-1} = \lambda \omega^n \otimes iI. \tag{7}$$

If M is compact, then the constant λ is determined by the bundle E and the Kähler form ω. Specifically, we can take the trace of both sides of (6) and integrate to get

$$\lambda = \frac{2\pi}{r\,\mathrm{Vol}(M)} \int_M c_1(E) \wedge [\star\omega].$$

Here, $c_1(E)$ denotes the first Chern class of E, which satisfies the equation $c_1(E) = \frac{1}{2\pi i}[\mathrm{tr}(F_A)]$.

If M is a Kähler manifold of real dimension 4 and $E \to M$ is a Hermitian vector bundle, then a connection on E is an anti-self-dual instanton if and only if it is Hermitian Yang-Mills with $\lambda = 0$. This is because the anti-self-dual 2-forms on M are precisely the $(1,1)$ forms that are orthogonal to ω. So Hermitian Yang-Mills connections are a generalization of ASD instantons to higher dimensions and to nonzero values of λ.

The relationship between HYM connections and G_2 instantons that I hinted at earlier is given by the following proposition.

Proposition 3.4. *Let (X, ω, Υ) be a Calabi-Yau 3-fold and let $M \xrightarrow{\pi} X$ be an S^1 bundle with connection η, giving a G_2 structure on M as in Proposition 2.3. Let $E \to X$ be a Hermitian vector bundle, and let A be a unitary connection on E, giving a unitary connection π^*A on the pullback bundle π^*E. Then π^*A is a G_2-instanton if and only if A is Hermitian Yang-Mills with $\lambda = 0$.*

Proof. The coassociative form ψ on M is given by

$$\psi = \pi^* \tfrac{\omega^2}{2} - \eta \wedge \pi^* \mathrm{Im}(\Upsilon).$$

So the G_2 instanton equation for π^*A (which has curvature π^*F_A) is

$$\pi^*\left(F_A \wedge \tfrac{\omega^2}{2}\right) - \eta \wedge \pi^*(F_A \wedge \mathrm{Im}(\Upsilon)) = 0.$$

This holds if and only if both $F_A \wedge \frac{\omega^2}{2} = 0$ and $F_A \wedge \mathrm{Im}(\Upsilon) = 0$. The former equation is equation (7) in the case $n = 3$ and $\lambda = 0$, and the latter is equivalent to the statement that F_A is $(1,1)$. ☐

3.2 The Deformed G_2 Instanton Equation

If M is a smooth manifold and $E \to M$ is a Hermitian vector bundle, we get an "adjoint" map $(\alpha \mapsto \alpha^\dagger) : \Omega^p(M, \mathrm{End}_{\mathbb{C}}(E)) \to \Omega^p(M, \mathrm{End}_{\mathbb{C}}(E))$. Pointwise, this is just the tensor of the identity map on $\Lambda^p T^*_x M$ with the adjoint map $\mathrm{End}_{\mathbb{C}}(E_x) \to \mathrm{End}_{\mathbb{C}}(E_x)$. Then $\Omega^p(M, \mathfrak{u}(E))$ can be characterized as the set of forms $\alpha \in \Omega^p(M, \mathrm{End}_{\mathbb{C}}(E))$ such that $\alpha^\dagger = -\alpha$. We also have a wedge product $\wedge : \Omega^p(M, \mathrm{End}_{\mathbb{C}}(E)) \times \Omega^q(M, \mathrm{End}_{\mathbb{C}}(E)) \to \Omega^{p+q}(M, \mathrm{End}_{\mathbb{C}}(E))$ coming from composition of endomorphisms. Together, these operations satisfy the following rule: If $\alpha \in \Omega^p(M, \mathrm{End}_{\mathbb{C}}(E))$ and $\beta \in \Omega^q(M, \mathrm{End}_{\mathbb{C}}(E))$, then

$$(\alpha \wedge \beta)^\dagger = (-1)^{pq} \beta^\dagger \wedge \alpha^\dagger.$$

From this fact, we see that, if $\alpha \in \Omega^p(M, \mathfrak{u}(E))$ with p even, then α^k is self-adjoint for k even and anti-self-adjoint for k odd.

Definition 3.5 (Deformed G_2 instanton)**.** Let (M, φ, ψ) be a manifold with G_2 structure, and let $E \to M$ be a Hermitian vector bundle. A unitary connection A on E is called a deformed G_2 instanton if

$$\frac{1}{6}F_A^3 + F_A \wedge \psi = 0. \tag{8}$$

By the preceding remarks, the left hand side of this equation is always in $\Omega^6(M, \mathfrak{u}(E))$.

Note that equation (8) is identical to the definition of a G_2 instanton (Definition 3.1), but with the addition of the F_A^3 term. So a deformed G_2 instanton A with $F_A^3 = 0$ is also a G_2 instanton. (This is always the case if A was pulled back from a connection on a manifold of dimension less than 6.)

If (M, φ, ψ) is a manifold with G_2 structure and $E \to M$ is a Hermitian vector bundle, let $\mathcal{Q} : \mathrm{Conn}(E) \to \Omega^6(M, \mathfrak{u}(E))$ be the map $\mathcal{Q}(A) = \frac{1}{6}F_A^3 + F_A \wedge \psi$. (Here, $\mathrm{Conn}(E)$ is the space of unitary connections on E.) By definition, a deformed G_2 instanton is an element of $\mathcal{Q}^{-1}(0)$. Recall that we can regard $\mathrm{Conn}(E)$ as an affine space over $\Omega^1(M, \mathfrak{u}(E))$, and that, if $A \in \mathrm{Conn}(E)$ and $\alpha \in \Omega^1(M, \mathfrak{u}(E))$, then $F_{A+\alpha} = F_A + d_A\alpha + \alpha \wedge \alpha$. It follows that, if $A \in \mathrm{Conn}(E)$, then the linearization of $\mathcal{Q}$ at A, which I denote $\delta_A\mathcal{Q} : \Omega^1(M, \mathfrak{u}(E)) \to \Omega^6(M, \mathfrak{u}(E))$, is given by

$$\delta_A\mathcal{Q}(\alpha) = \frac{1}{6}\big(d_A\alpha \wedge F_A^2 + F_A^2 \wedge d_A\alpha + F_A \wedge d_A\alpha \wedge F_A\big) + d_A\alpha \wedge \psi. \tag{9}$$

This simplifies if E is rank 1; see Section 4.1

To motivate Definition 3.5, I will show how deformed G_2 instantons are related to a more well-known type of connection, namely deformed Hermitian Yang-Mills connections. The definition of the latter can be derived as follows. Let (M, ω) be an n-complex-dimensional Kähler manifold, and let $E \to M$ be a Hermitian vector bundle with unitary connection A. Recall that A is a Hermitian Yang-Mills connection (Definition 3.3) if and only if F_A is $(1,1)$ and

$$n\omega^{n-1} \wedge F_A = \lambda\omega^n \otimes iI \tag{10}$$

for some $\lambda \in \mathbb{R}$. Note that $n\omega^{n-1} \wedge F_A$ and $\omega^n \otimes I$ are both terms in the expansion of $\Omega = (\omega \otimes I + F_A)^n$. Explicitly, the self-adjoint part of Ω is

$$\frac{\Omega + \Omega^\dagger}{2} = \omega^n \otimes I + \binom{n}{2}\omega^{n-2} \wedge F_A^2 + \cdots, \tag{11}$$

and the anti-self-adjoint part of Ω is

$$\frac{\Omega - \Omega^\dagger}{2} = n\omega^{n-1} \wedge F_A + \binom{n}{3}\omega^{n-3} \wedge F_A^3 + \cdots. \tag{12}$$

If we replace $\omega^n \otimes I$ with $\frac{\Omega+\Omega^\dagger}{2}$ and $n\omega^{n-1} \wedge F_A$ with $\frac{\Omega-\Omega^\dagger}{2}$ in equation (10), we obtain a "deformed" version of the HYM equation:

$$\frac{\Omega - \Omega^\dagger}{2} = i\lambda\frac{\Omega + \Omega^\dagger}{2}. \tag{13}$$

(In the "large volume limit", where ω is scaled up by a large factor, the sums in (11) and (12) can be approximated by their first terms, and thus equation (13) can be approximated by equation (10).) If we write $\lambda = \tan\theta$, then equation (13) becomes $e^{-i\theta}\Omega = e^{i\theta}\Omega^\dagger$, which states that $e^{-i\theta}\Omega$ is self-adjoint. This motivates the following definition.

Definition 3.6 (Deformed Hermitian Yang-Mills connection)**.** Let (M, ω) be an n-complex-dimensional Kähler manifold, let $E \to M$ be a Hermitian vector bundle, and let θ be some real number. A unitary connection A on E is called a deformed Hermitian Yang-Mills connection with phase $e^{i\theta}$ if F_A is $(1,1)$ and the form

$$e^{-i\theta}(\omega \otimes I + F_A)^n \in \Omega^{2n}(M, \mathrm{End}_{\mathbb{C}}(E))$$

is self-adjoint.

If M is compact, then θ is constrained by the vector bundle E, as with the λ parameter in the definition of a Hermitian Yang-Mills connection.

The relationship between deformed HYM connections and deformed G_2 instantons is the same as that between HYM connections and G_2 instantons (Proposition 3.4).

Proposition 3.7. *Let (X, ω, Υ) be a Calabi-Yau 3-fold and let $M \xrightarrow{\pi} X$ be an S^1 bundle with connection η, giving a G_2 structure on M as in Proposition 2.3. Let $E \to X$ be a Hermitian vector bundle, and let A be a unitary connection on E, giving a unitary connection π^*A on the pullback bundle π^*E. Then π^*A is a deformed G_2-instanton if and only if A is deformed HYM with phase 1.*

Proof. By the same argument as in Proposition 3.4, we see that π^*A is a deformed G_2 instanton if and only if $F_A \wedge \mathrm{Im}(\Upsilon) = 0$ and $\frac{1}{6}F_A^3 + F_A \wedge \frac{\omega^2}{2} = 0$. The first equation is equivalent to the statement that F_A is $(1,1)$. And the left hand side of the second equation is $\frac{1}{6}$ times the anti-self-adjoint part of $(\omega \otimes I + F_A)^n$, so the second equation is equivalent to the statement that $(\omega \otimes I + F_A)^n$ is self-adjoint. $\square$

Both deformed HYM connections and deformed G_2 instantons are primarily studied for rank-1 Hermitian vector bundles, or equivalently, principal $U(1)$ bundles. I have considered higher-rank bundles in this section to demonstrate that the definition of deformed G_2 instanton still makes sense for higher-rank bundles (and is related to deformed HYM connections and thence HYM connections in the same way as in the rank-1 case). But in most of the remainder of this paper, including the sections on Aloff-Wallach spaces, I will be considering deformed G_2 instantons on principal $U(1)$ bundles.

In the rank-1 case, several simplifications occur. If $E \to M$ is a Hermitian line bundle, then $\mathrm{End}_{\mathbb{C}}(E)$ is the trivial bundle $M \times \mathbb{C}$, and $\mathfrak{u}(E)$ is the trivial bundle $M \times i\mathbb{R}$. Analogously, if $P \to M$ is a $U(1)$ bundle, then $\mathrm{ad}(P) = M \times i\mathbb{R}$. If a connection is modified by $\alpha \in \Omega^1(M, i\mathbb{R})$, then its curvature (an element of $\Omega^2(M, i\mathbb{R})$) changes by $d\alpha$. A gauge transformation is given by a smooth function $\sigma : M \to U(1)$; this acts on connections by sending A to $A + \sigma^{-1}d\sigma$.

Note that $\sigma^{-1}d\sigma$ is a closed form (locally we can find $f : M \to i\mathbb{R}$ such that $\sigma = \exp f$, and then $\sigma^{-1}d\sigma = df$), so the curvature of a connection is invariant under gauge transformations. In fact, a pure imaginary 1-form can be written as $\sigma^{-1}d\sigma$ for some $\sigma : M \to U(1)$ if and only if it is closed with cohomology class equal to $2\pi i$ times some integral cohomology class.

For parallel G_2 manifolds, there is a theory of mirror symmetry, in which two G_2 manifolds M, W have fibrations $M \to B \leftarrow W$ to a shared 3-dimensional base manifold B whose fibers are coassociative 4-tori. It seems that there is a correspondence between deformed G_2 instantons on W and associative sections of $M \to B$ equipped with flat connections. If instead B is 4-dimensional and the fibrations $M \to B \leftarrow W$ have associative 3-tori as fibers, then there seems to be a correspondence between deformed G_2 instantons on W and coassociative sections of $M \to B$ equipped with anti-self-dual connections. [16]

4 Results on Deformed G_2 Instantons

4.1 The Associated G_2 Structure

Let (M, φ, ψ) be a manifold with G_2 structure, and let $P \to M$ be a principal $U(1)$ bundle. As in Section 3.2, we get a nonlinear differential operator $\mathcal{Q}$: $\mathrm{Conn}(P) \to \Omega^6(M, i\mathbb{R})$, defined by $\mathcal{Q}(A) = \frac{1}{6}F_A^3 + F_A \wedge \psi$, whose zeroes are the deformed G_2 instantons. Since even-degree complex-valued forms commute, the formula (9) for the linearization of $\mathcal{Q}$ at a connection A simplifies to

$$\delta_A \mathcal{Q}(\alpha) = d\alpha \wedge \left(\tfrac{1}{2}F_A^2 + \psi\right). \tag{14}$$

If A is a deformed G_2 instanton, then the 4-form $\frac{1}{2}F_A^2 + \psi$ appearing in (14) is, up to sign, the coassociative form of a new G_2 structure depending on A and φ. Specifically, we have the following theorem, citing [15].

Proposition 4.1. *Let A be a deformed G_2 instanton on the $U(1)$ bundle $P \to M$ with respect to the G_2 structure φ, ψ. Then the real-valued function $1 + \frac{1}{2}\langle F_A^2, \psi\rangle$ is nonvanishing. So we can define*

$$\tilde{\varphi}_A = \left|1 + \frac{1}{2}\langle F_A^2, \psi\rangle\right|^{-3/4} (I_{TM} + (-iF_A)^\sharp)^* \varphi.$$

Here $(-iF_A)^\sharp$ denotes the type $(1,1)$ tensor $v \mapsto (-iF_A(v, -))^\sharp$, and the asterisk denotes a pullback (or pointwise composition). Then $\tilde{\varphi}_A$ is a G_2 structure on M, with coassociative form

$$\tilde{\psi}_A = \pm\left(\frac{1}{2}F_A^2 + \psi\right),$$

where the $\pm$ is the sign of $1 + \frac{1}{2}\langle F_A^2, \psi\rangle$.

Proof. See [15, Section 3.3] and [15, Appendix C]. The key theorem is [15, Theorem C.1]. $\square$

So the linearization of $\mathcal{Q}$ at a deformed G_2 instanton A can be written as $\delta_A \mathcal{Q}(\alpha) = \pm(d\alpha \wedge \tilde{\psi}_A)$. Note that

- If A is flat, then $\tilde{\varphi}_A = \varphi$.
- If $F_A = F_{A'}$ (meaning A and A' differ by a closed form), then $\tilde{\varphi}_A = \tilde{\varphi}_{A'}$.
- If ψ is closed, then so is $\tilde{\psi}_A$. In other words, if φ is a coclosed G_2 structure, then so is $\tilde{\varphi}_A$.

In the coclosed case, we have the following theorem.

Proposition 4.2. *Let M be a manifold with coclosed G_2 structure φ, ψ. Let $P \to M$ be a principal $U(1)$ bundle, and let $\mathcal{Q}(A) = \frac{1}{6}F_A^3 + F_A \wedge \psi$ be the nonlinear differential operator whose zeroes are deformed G_2 instantons on P. Then if A is a deformed G_2 instanton on P, the linearization $\delta_A \mathcal{Q}$ belongs to an elliptic complex*

$$\Omega^0(M, i\mathbb{R}) \xrightarrow{d} \Omega^1(M, i\mathbb{R}) \xrightarrow{\delta_A \mathcal{Q}} \Omega^6(M, i\mathbb{R}) \xrightarrow{d} \Omega^7(M, i\mathbb{R}). \tag{15}$$

Moreover, if M is compact, the two middle cohomology groups of this complex have the same dimension, namely $b^1(M) + \varsigma(M, \tilde{\varphi}_A)$.

Proof. Note that (15) is essentially the canonical complex of the coclosed G_2 structure $\tilde{\varphi}_A$ (but acting on imaginary forms, and with the middle operator possibly negated). So we can apply the results of Section 2.1 to find that (15) is an elliptic complex with cohomology groups of the stated dimensions. □

4.2 Deformation Theory of Deformed G_2 Instantons

Let M be a compact manifold with coclosed G_2 structure φ, ψ, and let $P \to M$ be a principal $U(1)$ bundle. As before, let $\mathcal{Q}(A) = \frac{1}{6}F_A^3 + F_A \wedge \psi$ be the deformed G_2 instanton operator. Note that $\mathcal{Q}$ takes values in a single cohomology class, so if there are any deformed G_2 instantons, then $\mathcal{Q}$ takes values in $d\Omega^5(M, i\mathbb{R})$. I will denote by $\mathcal{D}(M, \varphi, P)$ the moduli space of deformed G_2-instantons on P modulo gauge transformations. Since the moduli space of all connections modulo gauge transformations is a metrizable topological space [15], $\mathcal{D}(M, \varphi, P)$ is also metrizable (and in particular is Hausdorff). Moreover, we have the following two results, proven in [15].

Proposition 4.3. *Suppose that A is a deformed G_2 instanton on P with the property that $\varsigma(M, \tilde{\varphi}_A) = 0$. Then the connected component of $[A]$ in $\mathcal{D}(M, \varphi, P)$ is homeomorphic to a torus of dimension $b^1(M)$.*

Proof. See [15]. A sketch of the proof is as follows. By Proposition 4.2, we have an elliptic complex (15) containing $\delta_A \mathcal{Q}$ in which both middle cohomology groups have dimension $b^1(M)$. This means that $\ker(\delta_A \mathcal{Q})$ consists only of closed 1-forms, while $\operatorname{im}(\delta_A \mathcal{Q})$ consists of all exact 6-forms, that is, $\operatorname{im}(\delta_A \mathcal{Q}) = d\Omega^5(M, i\mathbb{R})$. Let $d_0 : \Omega^0(M, i\mathbb{R}) \to \Omega^1(M, i\mathbb{R})$ be the exterior derivative. We can think of $\ker(d_0^*)$ as the tangent space of A within the moduli space of all connections, since it is the orthogonal complement of the infinitesimal gauge transformations $\mathrm{im}(d_0)$. Define $\mathcal{R} : \ker(d_0^*) \to d\Omega^5(M, i\mathbb{R})$ by $\mathcal{R}(\alpha) = \mathcal{Q}(A + \alpha)$. Since $\delta_0\mathcal{R} = \delta_A\mathcal{Q}$ is surjective with $b^1(M)$-dimensional kernel, the implicit function theorem for Banach spaces gives that $\mathcal{R}^{-1}(0)$ is a $b^1(M)$-dimensional manifold in the vicinity of 0, which means that $\mathcal{D}(M, \varphi, P)$ is a $b^1(M)$-dimensional manifold in the vicinity of $[A]$.

If η is any closed form in $\Omega^1(M, i\mathbb{R})$, then $A+\eta$ is also a deformed G_2 instanton, with $\tilde{\varphi}_{A+\eta} = \tilde{\varphi}_A$. So $\mathcal{D}(M, \varphi, P)$ is a $b^1(M)$-dimensional manifold in the vicinity of $[A+\eta]$. Note that $A+\eta$ is related to A by a gauge transformation if and only if η is an integral closed form times $2\pi i$. So the subset of $\mathcal{D}(M, \varphi, P)$ that we obtain by adding closed forms to A is equivalent to $H^1(M, \mathbb{R})/2\pi H^1(M, \mathbb{Z})$, which is a $b^1(M)$-dimensional torus. By comparing dimensions, we find that this torus is the whole connected component. □

Proposition 4.4. *Suppose that A is a deformed G_2 instanton on P, and suppose that either (i) the G_2 structure φ on M is parallel or nearly parallel, or (ii) $F_A \neq 0$ on a dense set of M. Then for generic nearby coclosed G_2 structures (φ', ψ') such that ψ' is in the same de Rham cohomology class as ψ, we have that for all connections A' nearby to A which are deformed G_2 instantons with respect to φ', the connected component of $[A'] \in \mathcal{D}(M, \varphi', P)$ is homeomorphic to a $b^1(M)$-dimensional torus.*

Proof. See [15]. □

As a corollary of Proposition 4.3, we have the following:

Proposition 4.5. *Suppose that the G_2 structure φ on M is nearly parallel, and that A is a flat connection on P (which is trivially a deformed G_2 instanton). Then $[A]$ is an isolated point in $\mathcal{D}(M, \varphi, P)$.*

Proof. Since φ is nearly parallel, we know that $b^1(M) = 0$ and (by Proposition 2.6) that $\varsigma(M, \varphi) = 0$. And since A is flat, we have $\tilde{\varphi}_A = \varphi$. So by Proposition 4.3, the connected component of $[A]$ in $\mathcal{D}(M, \varphi, P)$ is a 0-dimensional torus, that is, a point. □

4.3 A Chern-Simons Type Functional

In [14], the authors introduce a "functional of Chern-Simons type" which has deformed G_2 instantons as critical points. Recall that, if M is a smooth manifold, $E \to M$ a Hermitian vector bundle, and A_0, A_1 two unitary connections on E, the difference $\mathrm{tr}(F_{A_1}^k) - \mathrm{tr}(F_{A_0}^k) \in \Omega^{2k}(M, \mathbb{C})$, known to be exact by Chern-Weil theory, is given as the exterior derivative of the Chern-Simons transgression form $\mathrm{CS}_k(A_0, A_1) \in \Omega^{2k-1}(M, \mathbb{C})$. That is, $\mathrm{tr}(F_{A_1}^k) = \mathrm{tr}(F_{A_0}^k) + d\mathrm{CS}_k(A_0, A_1)$. If we write $A_1 = A_0 + \beta$ with $\beta \in \Omega^1(M, \mathfrak{u}(E))$, and define $A_s = A_0 + s\beta$ for $s \in [0, 1]$, then

$$\mathrm{CS}_k(A_0, A_1) = k \int_0^1 \mathrm{tr}(F_{A_s}^{k-1} \wedge \beta)\, ds.$$

Note that $F_{A_s} = F_{A_0} + s d_{A_0}\beta + s^2 \beta \wedge \beta$, so the integrand can be written as a polynomial in s with coefficients in $\Omega^{2k-1}(M, \mathbb{C})$. The integration simply replaces each power s^j with the constant $1/(j+1)$.

To define the functional, suppose that M is a compact manifold with coclosed G_2 structure φ, ψ, and $E \to M$ is a Hermitian vector bundle. Let π denote the projection $\pi : [0,1] \times M \to M$. If A_t is a smoothly varying unitary connection on E for $t \in [0,1]$, we get a unitary connection $\mathbf{A}$ on $\pi^* E$. We define

$$\mathcal{F}(A_0, A_1) = \int_{[0,1]\times M} \operatorname{tr}(\exp(F_{\mathbf{A}} + \pi^*\psi)), \tag{16}$$

where we are only integrating the degree-8 component. Thus equation (16) can also be written as

$$\mathcal{F}(A_0, A_1) = \int_{[0,1]\times M} \left(\frac{1}{24} \operatorname{tr}(F_{\mathbf{A}}^4) + \frac{1}{2} \operatorname{tr}(F_{\mathbf{A}}^2) \wedge \pi^*\psi \right). \tag{17}$$

As the notation suggests, this does not depend on the path from A_0 to A_1. Note that $\mathcal{F}(A_0, A_1)$ is real, since $F_{\mathbf{A}}^4$ and $F_{\mathbf{A}}^2$ are self-adjoint. And if A_0, A_1, A_2 are three connections on E, we have

$$\mathcal{F}(A_0, A_2) = \mathcal{F}(A_0, A_1) + \mathcal{F}(A_1, A_2)$$

by composing paths. In other words, the functionals $\mathcal{F}(A_0, -)$ and $\mathcal{F}(A_1, -)$ differ by a constant, hence have the same critical points.

Using Chern-Simons forms, we can put equation (17) into a form that refers only to M as opposed to $[0,1] \times M$. We start by noting that $\operatorname{tr}(F_{\mathbf{A}}^k) = \pi^* \operatorname{tr}(F_{A_0}^k) + d\mathrm{CS}_k(\pi^* A_0, \mathbf{A})$, with $\operatorname{tr}(F_{A_0}^4) = \operatorname{tr}(F_{A_0}^2) \wedge \psi = 0$ for dimension reasons. So equation (17) can be written as

$$\mathcal{F}(A_0, A_1) = \int_{[0,1]\times M} d\left(\frac{1}{24} \mathrm{CS}_4(\pi^* A_0, \mathbf{A}) + \frac{1}{2} \mathrm{CS}_2(\pi^* A_0, \mathbf{A}) \wedge \pi^*\psi \right),$$

using the fact that ψ is closed. By applying Stokes' theorem, and the fact that the pullback of a Chern-Simons form is the Chern-Simons form of the pullback connections, we obtain

$$\mathcal{F}(A_0, A_1) = \int_M \left(\frac{1}{24} \mathrm{CS}_4(A_0, A_1) + \frac{1}{2} \mathrm{CS}_2(A_0, A_1) \wedge \psi \right). \tag{18}$$

Note that, if A is a connection on E and $\beta \in \Omega^1(M, \mathfrak{u}(E))$, then $\mathrm{CS}_k(A, A + t\beta)$ is a polynomial in t whose t coefficient is $\operatorname{tr}(kF_A^{k-1} \wedge \beta)$. Using this fact and the formula in (18), we can compute

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathcal{F}(A_0, A + t\beta) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathcal{F}(A, A + t\beta) \\ &= \int_M \operatorname{tr}\left(\left(\frac{1}{6} F_A^3 + F_A \wedge \psi \right) \wedge \beta \right) \\ &= -\left\langle\!\!\left\langle \frac{1}{6} F_A^3 + F_A \wedge \psi, \star\beta \right\rangle\!\!\right\rangle, \end{aligned} \tag{19}$$

where $\langle\!\langle -,- \rangle\!\rangle$ is an inner product on $\Omega^6(M,\mathfrak{u}(E))$. We now easily see the following.

Proposition 4.6. *Let M be a compact manifold with coclosed G_2 structure φ, ψ, and let $E \to M$ be a Hermitian vector bundle. Define the functional $\mathcal{F}$, taking two unitary connections on E as arguments, as in (16), with alternative formulas given by (17) and (18). Then a connection A on E is a critical point of $\mathcal{F}$ if and only if A is a deformed G_2 instanton.*

Proof. The derivative of $\mathcal{F}(A_0,-)$ at A in the direction of β is given by equation (19). This is zero for all β if and only if $\frac{1}{6}F_A^3 + F_A \wedge \psi = 0$, which is the deformed G_2 instanton equation. $\square$

In the rank-1 case, where the linearization of the deformed G_2 instanton equation simplifies to (14), we can compute from (19) that the Hessian of $\mathcal{F}(A_0,-)$ at a connection A is

$$H_A(\alpha,\beta) = \int_M \left(\frac{1}{2}F_A^2 + \psi\right) \wedge d\alpha \wedge \beta = -\left\langle\!\!\left\langle \left(\frac{1}{2}F_A^2 + \psi\right) \wedge d\alpha, \star\beta \right\rangle\!\!\right\rangle,$$

for $\alpha, \beta \in \Omega^1(M, i\mathbb{R})$. If A is a deformed G_2 instanton, then we can write this as $H_A(\alpha,\beta) = (-1)^j \langle\!\langle \mathrm{curl}_{\tilde{\varphi}_A}(\alpha), \beta \rangle\!\rangle$, for some j depending on A. If $\alpha \in \Omega^1(M, i\mathbb{R})$ is a unit-norm eigenvector of $\mathrm{curl}_{\tilde{\varphi}_A}$ with $\mathrm{curl}_{\tilde{\varphi}_A}(\alpha) = \lambda\alpha$, then $H_A(\alpha,\alpha) = (-1)^j\lambda$. Since $\mathrm{curl}_{\tilde{\varphi}_A}$ has smooth eigenvectors of both positive and negative eigenvalues (see Proposition 2.10), H_A is always indefinite, and every deformed G_2 instanton is a saddle point of $\mathcal{F}(A_0,-)$.

Still in the rank-1 case, we can compute the Chern-Simons forms in (18) to get a very explicit formula for the functional $\mathcal{F}$. (One can also do this in the higher-rank case, but there will be many more terms.) If A_0 is a unitary connection on E and $\beta \in \Omega^1(M, i\mathbb{R})$, then

$$\begin{aligned} \mathcal{F}(A_0, A_0+\beta) = \int_M \Bigg(& \frac{1}{6}F_{A_0}^3 + \frac{1}{4}F_{A_0}^2 \wedge d\beta + \frac{1}{6}F_{A_0} \wedge (d\beta)^2 \\ & \frac{1}{24}(d\beta)^3 + F_{A_0} \wedge \psi + \frac{1}{2}d\beta \wedge \psi \Bigg) \wedge \beta. \end{aligned}$$

If A_0 happens to be flat, then this formula simplifies further, with four of the six terms vanishing.

5 Circle Subgroups of SU(3)

In this section, I show that circle subgroups of SU(3) are classified up to conjugation by rational numbers in the interval $[0,1]$. (I have not seen the classification stated in this form in the literature.)

For rational numbers $k, l, m \in \mathbb{Q}$ not all zero such that $k + l + m = 0$, we have the circle subgroup $T_{k,l,m} \subseteq \mathrm{SU}(3)$ defined by

$$T_{k,l,m} = \left\{ \begin{pmatrix} e^{ikt} & 0 & 0 \\ 0 & e^{ilt} & 0 \\ 0 & 0 & e^{imt} \end{pmatrix} : t \in \mathbb{R} \right\}.$$

Clearly, $T_{rk,rl,rm} = T_{k,l,m}$ for $r \in \mathbb{Q}^\times$, so we could require without loss of generality that k, l, m be integers. But in what follows, it will be useful to allow rational numbers. In particular, for $q \in \mathbb{Q} \cap [0, 1]$, I define $T_q = T_{1,q,-1-q}$; I claim that every circle subgroup of $\mathrm{SU}(3)$ is conjugate to T_q for exactly one $q \in \mathbb{Q} \cap [0, 1]$. The first step in proving this is the following lemma.

Proposition 5.1. *For $k, l, m \in \mathbb{Q}$ not all zero such that $k + l + m = 0$, we have $T_{k,l,m} \sim T_{l,k,m} \sim T_{k,m,l}$, where $\sim$ denotes a conjugacy relationship. Since two transpositions generate all of the symmetric group S_3, this implies that we can always permute the three indices without changing the conjugacy class.*

Proof. Note that conjugation by

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{SU}(3)$$

permutes the first two diagonal entries of a diagonal matrix, hence sends $T_{k,l,m}$ to $T_{l,k,m}$. And conjugation by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \in \mathrm{SU}(3)$$

permutes the last two diagonal entries of a diagonal matrix, hence sends $T_{k,l,m}$ to $T_{k,m,l}$. □

Now we can prove the following, which is the first half of the classification.

Proposition 5.2. *If K is any circle subgroup of $\mathrm{SU}(3)$, then K is conjugate to T_q for some $q \in \mathbb{Q} \cap [0, 1]$.*

Proof. Since $\mathrm{SU}(3)$ is a compact Lie group, we know that K is contained in some maximal torus of $\mathrm{SU}(3)$, and that all maximal tori are conjugate. The standard maximal torus of $\mathrm{SU}(3)$ is two-dimensional and is the image of the embedding $f : U(1) \times U(1) \to \mathrm{SU}(3), f(e^{i\theta}, e^{i\phi}) = \mathrm{diag}(e^{i\theta}, e^{i\phi}, e^{i(-\theta-\phi)})$. The circle subgroups of $U(1) \times U(1)$ correspond to rationally sloped lines through the origin in $\mathbb{R}^2$, and are mapped by f to subgroups of the form $T_{k,l,m}$. So $K \sim T_{k,l,m}$ for some $k, l, m \in \mathbb{Q}$ not all zero with $k + l + m = 0$.

It remains to show that $T_{k,l,m} \sim T_q$ for some $q \in \mathbb{Q} \cap [0, 1]$. There are two cases:

- If one of k, l, m is 0, then the other two must be nonzero and negatives of each other. By applying Proposition 5.1 and scaling, we find that $T_{k,l,m} \sim T_{1,0,-1} = T_0$.
- If k, l, m are all nonzero, we can assume that two are positive and one is negative. Then by applying Proposition 5.1 (moving the larger positive index to first position and the smaller positive index to second position) and scaling, we find that $T_{k,l,m} \sim T_{1,r,-1-r} = T_r$ for some $0 < r \leq 1$. $\square$

The last thing we need is the following.

Proposition 5.3. *If q, $\tilde{q}$ are distinct rational numbers in $\mathbb{Q} \cap [0, 1]$, then T_q is not conjugate to $T_{\tilde{q}}$.*

Proof. Recall that the Lie algebra $\mathfrak{su}(3)$ of SU(3) consists of traceless anti-Hermitian 3×3 matrices, and has the SU(3)-invariant inner product

$$\langle A, B \rangle = \mathrm{Re}(\mathrm{tr}(A^\dagger B)) = -\mathrm{Re}(\mathrm{tr}(AB)).$$

For each $q \in \mathbb{Q} \cap [0, 1]$, define $A_q \in \mathfrak{su}(3)$ by

$$A_q = \frac{1}{\sqrt{2}\sqrt{1 + q + q^2}} \begin{pmatrix} i & 0 & 0 \\ 0 & iq & 0 \\ 0 & 0 & i(-1 - q) \end{pmatrix},$$

so that $\pm A_q$ are the unit vectors tangential to T_q. We compute

$$\det(A_q) = \frac{iq(1 + q)}{2^{3/2}(1 + q + q^2)^{3/2}}.$$

If T_q and $T_{\tilde{q}}$ are conjugate, say $UT_qU^{-1} = T_{\tilde{q}}$ with $U \in \mathrm{SU}(3)$, then since conjugation by U is an isometry on $\mathfrak{su}(3)$, we must have $UA_qU^{-1} = \pm A_{\tilde{q}}$, hence $\det(A_q) = \pm \det(A_{\tilde{q}})$. In fact this must be $\det(A_q) = \det(A_{\tilde{q}})$, since both are nonnegative multiples of i. The function

$$x \mapsto \frac{x(1 + x)}{(1 + x + x^2)^{3/2}}.$$

is strictly increasing on $[0, 1]$ (which can be seen by taking the derivative of its square), so we conclude that $q = \tilde{q}$. $\square$

By combining propositions 5.2 and 5.3, we obtain

Proposition 5.4. *If K is a circle subgroup of* SU(3)*, then there is exactly one $q \in \mathbb{Q} \cap [0, 1]$ such that K is conjugate to T_q.*

6 Aloff-Wallach Spaces

An Aloff-Wallach space, named after the authors of [1], is a quotient of SU(3) by the right action of a circle subgroup. Since SU(3) is 8-dimensional, every Aloff-Wallach space is 7-dimensional. By Theorem 5.4, we need only consider the Aloff-Wallach spaces $X_q = \mathrm{SU}(3)/T_q$ with $q \in \mathbb{Q} \cap [0,1]$; every Aloff-Wallach space is isomorphic to exactly one of these as an SU(3)-homogeneous space.

Fix some $q \in \mathbb{Q} \cap [0,1]$. Write $q = l/k$ with k, l coprime integers, $k > 0$, and $0 \le l \le k$. Let $m = -k - l$. Then $T_q = T_{k,l,m}$, using the notation of Section 5. I will primarily use the variables k, l, m rather than q, as this leads to more symmetric-looking expressions. Note that the parametrization of T_q given by $t \mapsto \mathrm{diag}(e^{ikt}, e^{ilt}, e^{imt})$ is 2π-periodic, since k and l are coprime. Let $s = \sqrt{k^2 + l^2 + m^2}$, and define an orthonormal basis for $\mathfrak{su}(3)$ by

$$Y = \frac{1}{s}\begin{pmatrix} ik & 0 & 0 \\ 0 & il & 0 \\ 0 & 0 & im \end{pmatrix}$$

$$Z_1 = \frac{1}{s\sqrt{3}}\begin{pmatrix} i(l-m) & 0 & 0 \\ 0 & i(m-k) & 0 \\ 0 & 0 & i(k-l) \end{pmatrix}$$

$$Z_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad Z_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}$$

$$Z_4 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \qquad Z_5 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

$$Z_6 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad Z_7 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Let μ be the Maurer-Cartan form of SU(3), which is a left-invariant $\mathfrak{su}(3)$ valued 1-form on SU(3). Using the basis above, write $\mu = \eta \otimes Y + \sum_{j=1}^{7} \alpha_j \otimes Z_j$ with $\eta, \alpha_1, \ldots, \alpha_7 \in \Omega^1(\mathrm{SU}(3), \mathbb{R})$. Then $\eta, \alpha_1, \ldots, \alpha_7$ is the dual basis of the left translations of $Y, Z_1, \ldots, Z_7$. Since the left translation of Y points in the direction of the right action of T_q, we see that $\alpha_1 \ldots \alpha_7$ is a basis for the horizontal 1-forms on SU(3), viewing the latter as a fiber bundle over X_q. If we define

$$\beta_1 = \alpha_2 + i\alpha_3, \quad \beta_2 = \alpha_4 + i\alpha_5, \quad \beta_3 = \alpha_6 + i\alpha_7,$$

then

$$\mu = \begin{pmatrix} i\left(\frac{k}{s}\eta + \frac{l-m}{s\sqrt{3}}\alpha_1\right) & -\frac{1}{\sqrt{2}}\overline{\beta_3} & \frac{1}{\sqrt{2}}\beta_2 \\ \frac{1}{\sqrt{2}}\beta_3 & i\left(\frac{l}{s}\eta + \frac{m-k}{s\sqrt{3}}\alpha_1\right) & -\frac{1}{\sqrt{2}}\overline{\beta_1} \\ -\frac{1}{\sqrt{2}}\overline{\beta_2} & \frac{1}{\sqrt{2}}\beta_1 & i\left(\frac{m}{s}\eta + \frac{k-l}{s\sqrt{3}}\alpha_1\right) \end{pmatrix}$$

Let $A(t) = \mathrm{diag}(e^{ikt}, e^{ilt}, e^{imt})$ be the 2π-periodic parametrization of T_q. Using the identity $R^*_{A(t)}\mu = \mathrm{ad}(A(t)^{-1})\mu$ and the above expression for μ, we see that

$$\begin{gathered} R^*_{A(t)}\eta = \eta \qquad R^*_{A(t)}\alpha_1 = \alpha_1 \\ R^*_{A(t)}\beta_1 = e^{i(l-m)t}\beta_1 \qquad R^*_{A(t)}\beta_2 = e^{i(m-k)t}\beta_2 \qquad R^*_{A(t)}\beta_3 = e^{i(k-l)t}\beta_3. \end{gathered} \tag{20}$$

It follows that the following forms are invariant under the right action of T_q:

$$\begin{gathered} \alpha_1, \quad \frac{1}{2i}\overline{\beta_1} \wedge \beta_1 = \alpha_{23}, \quad \frac{1}{2i}\overline{\beta_2} \wedge \beta_2 = \alpha_{45}, \quad \frac{1}{2i}\overline{\beta_3} \wedge \beta_3 = \alpha_{67}, \\ \mathrm{Re}(\beta_1 \wedge \beta_2 \wedge \beta_3) = \alpha_{246} - \alpha_{257} - \alpha_{347} - \alpha_{356}, \\ \mathrm{Im}(\beta_1 \wedge \beta_2 \wedge \beta_3) = -\alpha_{357} + \alpha_{346} + \alpha_{256} + \alpha_{247}. \end{gathered}$$

Since these are also horizontal, they descend to forms on X_q. Fix $A, B, C, D \in \mathbb{R}^\times$ and define $\varphi \in \Omega^3(X_q)$ by

$$\varphi = D\alpha_1 \wedge (A^2\alpha_{23} + B^2\alpha_{45} + C^2\alpha_{67}) + ABC(\alpha_{246} - \alpha_{257} - \alpha_{347} - \alpha_{356}). \tag{21}$$

Comparing this to the formula (1) for the standard G_2 structure on $\mathbb{R}^7$, we see that φ defines a G_2 structure on X_q whose metric, coassociative form, and volume form are

$$\begin{aligned} g &= D^2\alpha_1^2 + A^2(\alpha_2^2 + \alpha_3^2) + B^2(\alpha_4^2 + \alpha_5^2) + C^2(\alpha_6^2 + \alpha_7^2) \\ \psi &= B^2C^2\alpha_{4567} + A^2C^2\alpha_{2367} + A^2B^2\alpha_{2345} \\ &\quad + ABCD\alpha_1 \wedge (\alpha_{357} - \alpha_{346} - \alpha_{256} - \alpha_{247}) \\ \mathrm{vol} &= A^2B^2C^2D\alpha_{1234567} \end{aligned} \tag{22}$$

Note that, if we negate two of A, B, C, the G_2 structure φ does not change. We can eliminate this redundancy by requiring $A, B > 0$. Then the space of G_2 structures that we are considering is one-to-one parametrized by $(\mathbb{R}_{>0})^2 \times (\mathbb{R}^\times)^2$. Note also that it is the sign of D that determines the orientation, and that negating φ corresponds to negating both C and D.

Using the Maurer-Cartan identity $d\mu = -\mu \wedge \mu$ (where the wedge product comes from matrix multiplication), we see that

$$
\begin{aligned}
d\eta &= \frac{1}{s}((l-m)\alpha_{23} + (m-k)\alpha_{45} + (k-l)\alpha_{67}) \qquad (23)\\
d\alpha_1 &= \frac{-\sqrt{3}}{s}(k\alpha_{23} + l\alpha_{45} + m\alpha_{67})\\
d\alpha_2 &= \frac{m-l}{s}\eta \wedge \alpha_3 + \frac{\sqrt{3}k}{s}\alpha_{13} - \frac{1}{\sqrt{2}}\alpha_{46} + \frac{1}{\sqrt{2}}\alpha_{57}\\
d\alpha_3 &= \frac{l-m}{s}\eta \wedge \alpha_2 - \frac{\sqrt{3}k}{s}\alpha_{12} + \frac{1}{\sqrt{2}}\alpha_{47} + \frac{1}{\sqrt{2}}\alpha_{56}\\
d\alpha_4 &= \frac{k-m}{s}\eta \wedge \alpha_5 + \frac{\sqrt{3}l}{s}\alpha_{15} + \frac{1}{\sqrt{2}}\alpha_{26} - \frac{1}{\sqrt{2}}\alpha_{37}\\
d\alpha_5 &= \frac{m-k}{s}\eta \wedge \alpha_4 - \frac{\sqrt{3}l}{s}\alpha_{14} - \frac{1}{\sqrt{2}}\alpha_{36} - \frac{1}{\sqrt{2}}\alpha_{27}\\
d\alpha_6 &= \frac{l-k}{s}\eta \wedge \alpha_7 + \frac{\sqrt{3}m}{s}\alpha_{17} - \frac{1}{\sqrt{2}}\alpha_{24} + \frac{1}{\sqrt{2}}\alpha_{35}\\
d\alpha_7 &= \frac{k-l}{s}\eta \wedge \alpha_6 - \frac{\sqrt{3}m}{s}\alpha_{16} + \frac{1}{\sqrt{2}}\alpha_{25} + \frac{1}{\sqrt{2}}\alpha_{34}.
\end{aligned}
$$

Using these formulas, one can compute $d\psi = 0$ and

$$
\begin{aligned}
d\varphi = & -\left(2\sqrt{2}ABC + \frac{\sqrt{3}D}{s}(C^2 l + B^2 m)\right)\alpha_{4567}\\
& -\left(2\sqrt{2}ABC + \frac{\sqrt{3}D}{s}(A^2 m + C^2 k)\right)\alpha_{2367}\\
& -\left(2\sqrt{2}ABC + \frac{\sqrt{3}D}{s}(B^2 k + A^2 l)\right)\alpha_{2345}\\
& -\frac{\sqrt{2}}{2}(A^2 + B^2 + C^2)D\alpha_1 \wedge (\alpha_{357} - \alpha_{346} - \alpha_{256} - \alpha_{247}).
\end{aligned}
$$

So, among the G_2 structures under consideration, all are coclosed, and none are parallel (since the last term in the formula for $d\varphi$ is always nonzero). The G_2 structure is nearly parallel with $d\varphi = \lambda\psi$ if and only if

$$
\begin{aligned}
2\sqrt{2}ABC + \frac{\sqrt{3}D}{s}(C^2 l + B^2 m) + \lambda B^2 C^2 &= 0\\
2\sqrt{2}ABC + \frac{\sqrt{3}D}{s}(A^2 m + C^2 k) + \lambda A^2 C^2 &= 0\\
2\sqrt{2}ABC + \frac{\sqrt{3}D}{s}(B^2 k + A^2 l) + \lambda A^2 B^2 &= 0\\
A^2 + B^2 + C^2 + \lambda\sqrt{2}ABC &= 0.
\end{aligned}
$$

By the proof in [6, Theorem 3.2], this system of equations for A, B, C, D has exactly 2 solutions in $(\mathbb{R}_{>0})^2 \times (\mathbb{R}^\times)^2$. Note that the fourth equation implies $\lambda C < 0$, so both nearly parallel G_2 structures have the same sign of C. However, they have different signs of D, hence opposite orientations.

If $q = 1$, which implies that $k = l = 1$, $m = -2$, and $s = \sqrt{6}$, then the two solutions are given explicitly by

$$A = B = \frac{6}{\sqrt{5}|\lambda|}, \quad C = D = \frac{-6\sqrt{2}}{5\lambda} \tag{24}$$

$$A = B = \frac{2}{|\lambda|}, \quad C = \frac{-2\sqrt{2}}{\lambda}, \quad D = \frac{2\sqrt{2}}{\lambda}. \tag{25}$$

Since X_q admits SU(3)-invariant nearly parallel G_2 structures, it in particular admits SU(3)-invariant positive Einstein metrics, and we have $b^1(X_q) = 0$.

If $0 < q < 1$, then by the analysis in [6], the 4-dimensional family of G_2 structures we are considering exhausts all SU(3)-invariant coclosed G_2 structures on X_q. In fact, this is also true if $q = 0$, as I show in the appendix (Proposition A.3).[2] However, if $q = 1$, there is a 10-dimensional space of SU(3)-invariant coclosed G_2 structures (Proposition A.4), of which the G_2 structures we are considering form a proper subset.

7 Deformed G_2 Instantons on X_q

Fix $q \in \mathbb{Q} \cap [0, 1]$, which determines $k, l, m \in \mathbb{Z}$ as in Section 6. Define an isomorphism χ from $U(1)$ to $T_q \subseteq \mathrm{SU}(3)$ by $\chi(z) = \mathrm{diag}(z^k, z^l, z^m)$. This makes SU(3) into a principal $U(1)$ bundle over X_q. For each $n \in \mathbb{Z}$, the homomorphism $\lambda_n : U(1) \to U(1)$ given by $\lambda_n(z) = z^n$ gives an associated bundle, which we denote $P_n \to X_q$. Formally, P_n is the quotient of $\mathrm{SU}(3) \times U(1)$ by the $U(1)$-action $(a, w)z = (a\chi(z), z^{-n}w)$.

Note that P_0 is just the trivial $U(1)$-bundle over X_q, and that P_1 is SU(3) itself. In general, for $n \neq 0$, we can identify P_n with the quotient of SU(3) by the order-$|n|$ cyclic subgroup of T_q, with $U(1)$-action given by $[a]z = [a\chi(z^{1/n})]$. Specifically, the identification sends the equivalence class of $(a, w) \in \mathrm{SU}(3) \times U(1)$ to $[a\chi(w^{1/n})]$.

There is a "canonical" connection A_n on P_n; if $n = 0$ this is the trivial flat connection, and if $n \neq 0$ this is

$$A_n = \frac{in}{s}\eta,$$

where η is the 1-form on SU(3) from Section 6 (which is invariant under the right action of T_q and thus descends to a 1-form on P_n). The curvature of A_n is given by

$$F_n = \frac{in}{s^2}((l - m)\alpha_{23} + (m - k)\alpha_{45} + (k - l)\alpha_{67}). \tag{26}$$

[2] This implies that some occurrences in [2] of the condition "$k \neq \pm l, l \neq \pm m, m \neq \pm k$" could be replaced by "$k \neq l, l \neq m, m \neq k$".

We can obtain all connections on P_n by taking $A_n + \beta$ where β is a pure imaginary 1-form on X_q. Deformed G_2 instantons on P_n for different n are related by the following result.

Proposition 7.1. *Let n be a nonzero integer, let β be a pure imaginary 1-form on X_q, and let φ be any G_2 structure on X_q. Then $A_n + \beta$ is a deformed G_2 instanton on P_n with respect to φ if and only if $A_1 + \frac{1}{n}\beta$ is a deformed G_2 instanton on P_1 with respect to $|n|^{-3/2}\varphi$.*

Proof. Let ψ be the coassociative form of φ; then the coassociative form of $|n|^{-3/2}\varphi$ is $|n|^{-2}\psi = \frac{1}{n^2}\psi$. Note that the curvature of $A_n + \beta$ is $F_n + d\beta$, while the curvature of $A_1 + \frac{1}{n}\beta$ is $F_1 + \frac{1}{n}d\beta = \frac{1}{n}(F_n + d\beta)$. So the deformed G_2 instanton equation of $A_1 + \frac{1}{n}\beta$ with respect to $|n|^{-3/2}\varphi$ is

$$\frac{1}{6}\frac{1}{n^3}(F_n + d\beta)^3 + \frac{1}{n}(F_n + d\beta) \wedge \frac{1}{n^2}\psi = 0.$$

Multiplying by n^3 on both sides, we obtain the deformed G_2 instanton equation of $A_n + \beta$ with respect to φ. □

We will consider all modifications of A_n by SU(3)-invariant 1-forms on X_q. These are given by the following theorem. (See also Appendix A, which looks at SU(3)-invariant forms of all degrees.)

Proposition 7.2. *If $q \neq 1$, then X_q has a 1-dimensional space of* SU(3)*-invariant 1-forms, consisting of constant multiples of α_1. On the other hand, X_1 has a 3-dimensional space of* SU(3)*-invariant 1-forms, spanned by $\alpha_1, \alpha_6, \alpha_7$.*

Proof. Note that the left-invariant 1-forms on SU(3) which are horizontal with respect to the projection to X_q are given by $\mathbb{R}$-linear combinations of $\alpha_1, \dots, \alpha_7$. If we additionally require that the 1-form be invariant under the right action of T_q, this forces the coefficients of $\alpha_2, \dots, \alpha_7$ to be 0, by (20), assuming $q \neq 1$. If $q = 1$, then $k - l = 0$, so α_6, α_7 are T_q-invariant, and only the coefficients of $\alpha_2, \dots, \alpha_5$ are forced to be 0. □

We will make use of the full three-dimensional space of invariant 1-forms on X_1 in Section 8. For now, we consider the following connections on P_n, which are parametrized by $b \in \mathbb{R}$ and are well-defined on every X_q:

$$A_{n,b} = A_n + \frac{ib}{s\sqrt{3}}\alpha_1.$$

The curvature (i.e. exterior derivative) of $A_{n,b}$ is

$$F_{n,b} = \frac{i}{s^2}((l-m)n - kb)\alpha_{23} + \frac{i}{s^2}((m-k)n - lb)\alpha_{45} + \frac{i}{s^2}((k-l)n - mb)\alpha_{67}.$$

Write this as $F_{n,b} = ic_1\alpha_{23} + ic_2\alpha_{45} + ic_3\alpha_{67}$. Fix $(A, B, C, D) \in (\mathbb{R}_{>0})^2 \times (\mathbb{R}^\times)^2$ and let φ, ψ be the G_2 structure on X_q given in (21), (22). Note that

$$\frac{1}{6}F_{n,b}^3 + F_{n,b} \wedge \psi = i(-c_1c_2c_3 + c_1B^2C^2 + c_2A^2C^2 + c_3A^2B^2)\alpha_{234567}.$$

So $A_{n,b}$ is a deformed G_2 instanton with respect to φ if and only if

$$-c_1c_2c_3 + c_1B^2C^2 + c_2A^2C^2 + c_3A^2B^2 = 0.$$

If we expand out c_1, c_2, c_3 and then multiply by s^6 on both sides, this becomes the following cubic equation in b.

$$\begin{aligned}
klmb^3 &+ n(l-m)(m-k)(k-l)b^2 \\
&+ \big(-9n^2klm - s^4\big(kB^2C^2 + lA^2C^2 + mA^2B^2\big)\big)b \\
&+ \big(-n^3(l-m)(m-k)(k-l) \\
&\qquad + s^4n\big((l-m)B^2C^2 + (m-k)A^2C^2 + (k-l)A^2B^2\big)\big) \\
&= 0.
\end{aligned} \tag{27}$$

The following summarizes what we know so far.

Proposition 7.3. *The connection $A_{n,b}$ on the $U(1)$-bundle $P_n \to X_q$ is a deformed G_2 instanton with respect to the G_2 structure defined by A, B, C, D if and only if the polynomial equation (27) holds. If $q \neq 1$, this takes into account every* $\mathrm{SU}(3)$*-invariant connection on P_n, as well as every* $\mathrm{SU}(3)$*-invariant coclosed G_2 structure on X_q.*

Note that equation (27) does not depend on D, nor on the sign of C. For certain values of q, n, A, B, C, some of the coefficients in (27) can vanish, giving rise to the following result.

Proposition 7.4. *The degree of the polynomial in equation (27) varies as follows.*

1. *If $q \neq 0$, then the polynomial has degree 3 and thus has 1, 2, or 3 solutions.*
2. *If $q = 0$ and $n \neq 0$, then the polynomial has degree 2 and thus has 0, 1, or 2 solutions.*
3. *If $q = 0$, $n = 0$, and $A^2 \neq C^2$, then the polynomial has degree 1 and thus has a unique solution.*
4. *If $q = 0$, $n = 0$, and $A^2 = C^2$, then the polynomial is 0, so every $b \in \mathbb{R}$ is a solution.*

Furthermore, if $n = 0$, then $b = 0$ is always a solution. (So in particular, the unique solution in case 3 above is $b = 0$.)

Proof. The coefficient of b^3 in equation (27) is klm, which is zero if and only if $q = 0$. If $q = 0$, then $k = 1$, $l = 0$, $m = -1$, so the coefficient of b^2 is $-2n$, which is zero if and only if $n = 0$. If $q = 0$ and $n = 0$, then the coefficient of b is $4B^2(A^2 - C^2)$, which is zero if and only if $A^2 = C^2$. If $q = 0$ and $n = 0$ and $A^2 = C^2$, then not only the coefficient of b but also the constant coefficient is zero. This proves all of the claims concerning the degree.

If $n = 0$, then the constant coefficient in (27) is zero, so $b = 0$ is a solution. This fact can also be seen by noting that $A_{0,0}$ is flat and thus is trivially a deformed G_2 instanton for any G_2 structure. $\square$

The cases that require more attention are the cubic and quadratic cases. These are analyzed in the next three subsections. Note that, by Proposition 7.1, we need only consider the case where $n \in \{0, 1\}$; solutions with $n \notin \{0, 1\}$ are related to solutions with $n = 1$ by scaling b and scaling the G_2 structure. This can also be seen from the fact that the polynomial in (27) is homogeneous in the variables n, b, A^2, B^2, C^2.

7.1 $q = 0,\ n = 1$

If $q = 0$, we have $k = 1$, $l = 0$, $m = -1$, $s = \sqrt{2}$. If moreover $n = 1$, then equation (27) becomes

$$b^2 - 2B^2(A^2 - C^2)b - 1 - 2B^2C^2 + 4A^2C^2 - 2A^2B^2 = 0.$$

(I have divided by -2 to make the polynomial monic.) The discriminant of this quadratic polynomial is

$$\Delta = 4B^4(A^2 - C^2)^2 + 4 + 8B^2C^2 - 16A^2C^2 + 8A^2B^2 \tag{28}$$

and the solutions are

$$b = B^2(A^2 - C^2) \pm \frac{1}{2}\sqrt{\Delta} \tag{29}$$

by the quadratic formula. This is summarized in the following result.

Proposition 7.5. *On the Aloff-Wallach space X_0 with G_2 structure determined by A, B, C, D, the number of* $\mathrm{SU}(3)$*-invariant deformed G_2 instantons on $P_1 = \mathrm{SU}(3) \to X_0$ is determined by the sign of Δ, defined as in* (28)*. If $\Delta < 0$, there are none, if $\Delta = 0$, there is one, and if $\Delta > 0$ there are two. If $\Delta \geq 0$, then the* $\mathrm{SU}(3)$*-invariant deformed G_2 instantons are given by $A_{1,b}$ with b as in* (29).

The equation $\Delta = 0$ defines a smooth surface in $\mathbb{R}^3$. As the G_2 structure crosses this surface, the two $\mathrm{SU}(3)$-invariant deformed G_2 instantons merge into a single deformed G_2 instanton and then disappear.

7.2 $q \neq 0,\ n = 0$

If $n = 0$, then equation (27) becomes

$$klmb^3 - s^4(kB^2C^2 + lA^2C^2 + mA^2B^2)b = 0.$$

We also assume that $q \neq 0$, so $k, l > 0$ and $m = -k - l < 0$. Since $klm \neq 0$, the solutions to the equation above are $b = 0$, along with the solutions of

$$b^2 = \frac{s^4}{klm}(kB^2C^2 + lA^2C^2 + mA^2B^2).$$

Since $s^4/klm < 0$, we have the following result.

Proposition 7.6. *On the Aloff-Wallach space X_q with $q \notin \{0, 1\}$, equipped with the G_2 structure determined by A, B, C, D, the number of* $\mathrm{SU}(3)$*-invariant deformed G_2 instantons on the trivial $U(1)$ bundle $P_0 \to X_q$ is determined by the sign of*

$$\Delta = kB^2C^2 + lA^2C^2 + mA^2B^2 = (kB^2 + lA^2)C^2 - (k + l)A^2B^2.$$

If $\Delta \geq 0$, then there is a unique $\mathrm{SU}(3)$*-invariant deformed G_2 instanton, namely the trivial flat connection. If $\Delta < 0$, then there are three, namely the trivial flat connection along with $A_{0,b}$ with $b = \pm s^2\sqrt{\Delta/klm}$.*

If $q = 1$, then this result still holds, except that we are not considering all $\mathrm{SU}(3)$*-invariant connections, only those of the form $A_{0,b}$ with $b \in \mathbb{R}$.*

Note that the condition $\Delta < 0$ can also be written as

$$C^2 < \frac{(k + l)A^2B^2}{kB^2 + lA^2}.$$

The right-hand side is always a positive real number, so by varying C, both $\Delta \geq 0$ and $\Delta < 0$ are achievable. This implies that, on every X_q with $q \notin \{0, 1\}$, there are some invariant G_2 structures in which the trivial $U(1)$ bundle has 1 invariant deformed G_2 instanton and others in which it has 3.

7.3 $q \neq 0$, $n = 1$

This is the most difficult case to analyze, because equation (27) remains a general cubic equation. (Of course, one could always apply the cubic formula, but that would not be very enlightening.) Note that the coefficients of b^3 and b^2 in (27) (with $n = 1$) depend only on the Aloff-Wallach space X_q, while the coefficients of b^1 and b^0 also depend on the G_2 structure. In fact, the following lemma proves that these latter coefficients can be made to take on any values by varying A, B, C.

Proposition 7.7. *Given $k, l, m \in \mathbb{Z}$ not all zero such that $k + l + m = 0$, the map $f : (\mathbb{R}_{>0})^3 \to \mathbb{R}^2$ given by*

$$\begin{aligned} f(A, B, C) = (&kB^2C^2 + lA^2C^2 + mA^2B^2, \\ &(l - m)B^2C^2 + (m - k)A^2C^2 + (k - l)A^2B^2) \end{aligned}$$

is surjective.

Proof. Consider the matrix

$$P = \begin{pmatrix} k & l & m \\ l-m & m-k & k-l \end{pmatrix}.$$

Note that the rows of P are nonzero and orthogonal to each other, so P has rank 2 and is surjective when viewed as a map $\mathbb{R}^3 \to \mathbb{R}^2$. Moreover, the kernel of P consists of scalar multiples of $(1,1,1)$. So, if $(u,v) \in \mathbb{R}^2$ is arbitrary, there exists $(a,b,c) \in \mathbb{R}^3$ such that $ka + lb + mc = u$ and $(l-m)a + (m-k)b + (k-l)c = v$. And by adding a scalar multiple of $(1,1,1)$, we can get $a, b, c > 0$. We can then define $A = (bc/a)^{1/4}, B = (ac/b)^{1/4}, C = (ab/c)^{1/4}$, so that

$$B^2C^2 = a, \quad A^2C^2 = b, \quad A^2B^2 = c,$$

and hence $f(A,B,C) = (u,v)$. $\square$

By varying only the b^1 and b^0 coefficients of a cubic polynomial, it can be made to have 1, 2, or 3 real roots. To see this, note that its derivative (a quadratic polynomial) can be made to have 2 real roots by varying the b^1 coefficient. Then by varying the b^0 coefficient, the cubic can be made to have 1, 2, or 3 real roots. Thus we obtain the following theorem.

Proposition 7.8. *On the Aloff-Wallach space X_q with $q \neq 0$ and G_2 structure determined by A, B, C, D, the $U(1)$ bundle P_n with $n \neq 0$ has either 1, 2, or 3 deformed G_2 instantons of the form $A_{n,b}$ with $b \in \mathbb{R}$, and all three possibilities can be obtained by varying A, B, C. If $q \neq 1$, then we are taking into account all invariant coclosed G_2 structures on X_q as well as all invariant connections on P_n.*

7.4 Isolatedness

Since $b^1(X_q) = 0$, Propositions 4.3 and 4.4 give conditions for a deformed G_2 instanton on X_q to be isolated in the moduli space. Indeed, Proposition 4.5 demonstrates that some of the deformed G_2 instantons we have seen in this section are isolated. Specifically, on every X_q, some of the G_2 structures we are considering are nearly parallel, and for these G_2 structures, the flat connection $A_{0,0}$ on the trivial bundle $P_0 \to X_q$ is an isolated deformed G_2 instanton. More generally, if any of the deformed G_2 instantons A have the property that their associated G_2 structure $\tilde{\varphi}_A$ (which is necessarily SU(3)-invariant since both φ and F_A are) is nearly parallel, then A is isolated.

The connections $A_{n,b}$ with $(n,b) \neq (0,0)$ are not flat. Since their curvatures are SU(3)-invariant, this implies that their curvatures are nonzero everywhere. So by Proposition 4.4, whenever $A_{n,b}$ with $(n,b) \neq (0,0)$ is a deformed G_2 instanton, we know that for generic nearby coclosed G_2 structures whose coassociative form is in the same cohomology class, deformed G_2 instantons near $A_{n,b}$ are isolated. In fact, $b^4(X_q) = 0$ [9], so all coclosed G_2 structures on X_q have cohomology class 0.

Some of the deformed G_2 instantons we have seen are clearly not isolated: specifically, in Proposition 7.4, we saw that over X_0 with a G_2 structure with $A^2 = C^2$, every connection $A_{0,b}$ on $P_0 \to X_0$ is a deformed G_2 instanton. This gives the following theorem.

Proposition 7.9. *On the Aloff-Wallach space X_0, the G_2 structures φ as in (21) with $A^2 = C^2$ satisfy $\varsigma(X_0, \varphi) > 0$.*

Proof. If we had $\varsigma(X_0, \varphi) = 0$, then $A_{0,0}$ would be isolated in the moduli space $\mathcal{D}(X_0, \varphi, P_0)$, by Proposition 4.3 and the fact that $\tilde{\varphi}_{A_{0,0}} = \varphi$. But it is not. $\square$

Here we see an interesting way of computing the invariant ς of a G_2 structure (or rather, proving that it is nonzero) that uses deformed G_2 instantons.

In Section 8, we will see that some of the deformed G_2 instantons on X_1 that we have found are also not isolated, instead belonging to a 2-sphere of deformed G_2 instantons.

8 Deformed G_2 Instantons on X_1

Although X_1 was not excluded from the analysis in Section 7, there are more SU(3)-invariant 1-forms on X_1 than on the other Aloff-Wallach spaces. In addition to scalar multiples of α_1, we have all linear combinations of $\alpha_1, \alpha_6, \alpha_7$. So the general form of an invariant connection on $P_n \to X_1$ is

$$A_{n,\vec{c}} = A_n + i(c_1\alpha_1 + c_6\alpha_6 + c_7\alpha_7)$$

with $\vec{c} = (c_1, c_6, c_7) \in \mathbb{R}^3$. The connections $A_{n,b}$ considered in Section 7 are included in the ones above by setting $c_1 = \frac{b}{3\sqrt{2}}, c_6 = c_7 = 0$. The curvature of $A_{n,\vec{c}}$ is

$$\begin{aligned} F_{n,\vec{c}} = &\frac{in}{2}(\alpha_{23} - \alpha_{45}) + \frac{ic_1}{\sqrt{2}}(2\alpha_{67} - \alpha_{23} - \alpha_{45}) \\ &+ \frac{ic_6}{\sqrt{2}}(-2\alpha_{17} - \alpha_{24} + \alpha_{35}) + \frac{ic_7}{\sqrt{2}}(2\alpha_{16} + \alpha_{25} + \alpha_{34}). \end{aligned}$$

X_1 also differs from the other Aloff-Wallach spaces in that it has a larger space of invariant coclosed G_2 structures, which is 10-dimensional instead of 4-dimensional (Proposition A.4). However, I have not found a way of parametrizing the full space of invariant coclosed G_2 structures (and this would likely lead to unwieldy equations in any case), so I will consider only the G_2 structures parametrized by A, B, C, D as in (21).[3] Let φ, ψ be such a G_2 structure. Expanding the deformed G_2 instanton equation $\frac{1}{6}F_{n,\vec{c}}^3 + F_{n,\vec{c}} \wedge \psi = 0$ yields three

[3]Some of the other invariant coclosed G_2 structures on X_1 can be obtained by applying symmetries of X_1, specifically the right action of SU(2) on X_1 that I discuss in Section 8.1.

equations:

$$(4ABCD - 4A^2B^2 - n^2 + 2c_1^2 + 2c_6^2 + 2c_7^2)c_6 = 0 \tag{30}$$

$$(4ABCD - 4A^2B^2 - n^2 + 2c_1^2 + 2c_6^2 + 2c_7^2)c_7 = 0 \tag{31}$$

$$\begin{aligned}&\sqrt{2}n(A^2 - B^2)C^2 \\ &\quad +(2B^2C^2 + 2A^2C^2 - 4A^2B^2 - n^2 + 2c_1^2 + 2c_6^2 + 2c_7^2)c_1 = 0.\end{aligned} \tag{32}$$

Note that D makes an appearance now, whereas it did not in Section 7.

If $c_6 = c_7 = 0$, then the connection is one of the ones considered in Section 7, and we can apply the results of that section. In particular, on the bundle $P_0 \to X_1$, there are either 1 or 3 deformed G_2 instantons with $c_6 = c_7 = 0$ (depending on the G_2 structure), which are given by $c_1 = 0$ and

$$c_1 = \pm\sqrt{2A^2B^2 - B^2C^2 - A^2C^2}. \tag{33}$$

(See Proposition 7.6.) And on the bundle $P_n \to X_1$ with $n \neq 0$, there are either 1, 2, or 3 deformed G_2 instantons with $c_6 = c_7 = 0$, with all three possibilities obtainable by varying the G_2 structure. (See Proposition 7.8.)

If $c_6 \neq 0$ or $c_7 \neq 0$, then by equation (30) or (31), a deformed G_2 instanton satisfies

$$4ABCD - 4A^2B^2 - n^2 + 2c_1^2 + 2c_6^2 + 2c_7^2 = 0. \tag{34}$$

Substituting this into equation (32), we obtain

$$\sqrt{2}n(A^2 - B^2)C^2 + (2B^2C^2 + 2A^2C^2 - 4ABCD)c_1 = 0. \tag{35}$$

Conversely, any connection satisfying (34) and (35) is a deformed G_2 instanton. So the invariant deformed G_2 instantons on $P_n \to X_1$ consist of solutions to (34) and (35), possibly with some additional isolated points along the line $c_6 = c_7 = 0$. This is expressed in the following theorems.

Proposition 8.1. *Equip X_1 with the G_2 structure in (21), and define the following:*

$$\begin{aligned}R &= 2A^2B^2 - 2ABCD \\ \Delta &= 2A^2B^2 - B^2C^2 - A^2C^2\end{aligned}$$

The the space of $\mathrm{SU}(3)$-invariant deformed G_2 instantons on the trivial $U(1)$ bundle $P_0 \to X_1$ is as follows:

1. *If $R \leq 0$ and $\Delta \leq 0$, we get a single solution $c_1 = c_6 = c_7 = 0$.*
2. *If $R \leq 0$ and $\Delta > 0$, we get three solutions $c_1 \in \{0, \pm\sqrt{\Delta}\}, c_6 = c_7 = 0$.*
3. *If $R > 0$ and $\Delta \leq 0$, we get a circle of solutions $c_6^2 + c_7^2 = R, c_1 = 0$ along with the point $c_1 = c_6 = c_7 = 0$.*
4. *If $R > 0$, $\Delta > 0$, and $R \neq \Delta$, we get a circle of solutions $c_6^2 + c_7^2 = R, c_1 = 0$, along with the three points $c_1 \in \{0, \pm\sqrt{\Delta}\}, c_6 = c_7 = 0$.*

5. *If $R > 0$ and $R = \Delta$, we get a sphere of solutions $c_1^2 + c_6^2 + c_7^2 = R$, along with the point $c_1 = c_6 = c_7 = 0$.*

Moreover, R, Δ can be made to take on any pair of values by varying the G_2 structure, so all five cases are possible.

Proof. The last remark is easy to prove: if we are given prescribed values $R_0, \Delta_0 \in \mathbb{R}$, set $A, B > 0$ so that $2A^2B^2 > R_0, \Delta_0$. Then set $C > 0$ so that $2A^2B^2 - B^2C^2 - A^2C^2 = \Delta_0$, and lastly set $D > 0$ so that $2A^2B^2 - 2ABCD = R_0$.

For the five cases, note that equations (34) and (35) can be written as $c_1^2 + c_6^2 + c_7^2 = R$ and $(R - \Delta)c_1 = 0$. If $R > 0$ and $R \neq \Delta$, the solutions to these two equations form a circle $c_6^2 + c_7^2 = R, c_1 = 0$. If $R > 0$ and $R = \Delta$, then we get the whole sphere $c_1^2 + c_6^2 + c_7^2 = R$. If $R \leq 0$, there are no solutions to (34), (35) except possibly the trivial solution $c_1 = c_6 = c_7 = 0$.

In addition to solutions of (34), (35), we also get some isolated points on the line $c_6 = c_7 = 0$. If $\Delta \leq 0$, there is only the trivial solution $c_1 = 0$, but if $\Delta > 0$, we also get solutions $c_1 = \pm\sqrt{\Delta}$, as noted in equation (33). □

Note that, in the setting of Proposition 8.1, there are three types of singularities that can occur as we vary the G_2 structure. As Δ goes from negative to positive, the trivial flat deformed G_2 instanton splits into three deformed G_2 instantons (which can be thought of as a point along with a 0-sphere). As R goes from negative to positive, the trivial flat deformed G_2 instanton splits into a point along with a 1-sphere. And if $R, \Delta > 0$, then as we go from $\Delta < R$ to $\Delta > R$ (so that the 0-sphere goes from being smaller than the 1-sphere to being larger), at the moment when $\Delta = R$, the 0-sphere and the 1-sphere become respectively the poles and equator of a 2-sphere.

While Proposition 8.1 concerns the trivial bundle P_0, the following theorem concerns the bundle $P_1 = \mathrm{SU}(3) \to X_1$. (By Proposition 7.1, the invariant deformed G_2 instantons on P_n with $n \neq 0$ are equivalent to those on P_1 by scaling c_1, c_6, c_7 and scaling the G_2 structure.)

Proposition 8.2. *Equip X_1 with the G_2 structure in (21), and define*

$$\begin{aligned} R &= \tfrac{1}{2} + 2A^2B^2 - 2ABCD \\ S &= B^2C^2 + A^2C^2 - 2ABCD \\ T &= \tfrac{\sqrt{2}}{2}(A^2 - B^2)C^2. \end{aligned}$$

Then the space of $\mathrm{SU}(3)$-invariant deformed G_2 instantons on $P_1 \to X_1$ is as follows:

1. *If $R > 0$ and $S = T = 0$, there is a 2-sphere $c_1^2 + c_6^2 + c_7^2 = R$, along with the point $c_1 = c_6 = c_7 = 0$.*

2. *If $R > 0$, $S \neq 0$, and $T^2 < RS^2$, there is a 1-sphere*

$$c_1 = \frac{-T}{S}, \quad c_6^2 + c_7^2 = R - \frac{T^2}{S^2}$$

along with 1, 2, or 3 points on the line $c_6 = c_7 = 0$.

3. *Otherwise (meaning* $R \leq 0$, *or* $R > 0$ *and* $(S,T) \neq (0,0)$ *and* $T^2 \geq RS^2$*), there are 1, 2, or 3 points on the line* $c_6 = c_7 = 0$, *and nothing else.*

Moreover, the deformed G_2 *instantons with* $c_6 = c_7 = 0$ *are given by the roots of the depressed cubic equation* $c_1^3 + (S-R)c_1 + T = 0$.

Proof. Note that equations (34) and (35) can be written as $c_1^2 + c_6^2 + c_7^2 = R$ and $T + Sc_1 = 0$. If $R > 0$, then the first equation describes a sphere of radius $\sqrt{R}$.

1. If moreover $S = T = 0$, then the second equation is always true, so the intersection is the whole sphere.

2. If instead $S \neq 0$, then the second equation describes a plane $c_1 = -T/S$, which intersects the sphere transversely if and only if $T^2/S^2 < R$, or equivalently $T^2 < RS^2$.

These are the only cases in which equations (34) and (35) give solutions off the line $c_6 = c_7 = 0$.

As for the deformed G_2 instantons on the line $c_6 = c_7 = 0$, note that in this case equations (30) and (31) are trivially satisfied, while equation (32) becomes $2T + (2S - 2R + 2c_1^2)c_1 = 0$, which can also be written as $c_1^3 + (S-R)c_1 + T = 0$. □

All of the cases in Proposition 8.2 are indeed possible:

Proposition 8.3. *The parameters* R, S, T *defined in Proposition 8.2 can be made to take on any triple of values by varying the* G_2 *structure.*

Proof. Suppose that we are given the prescribed values $R, S, T \in \mathbb{R}$. Set $u \in \mathbb{R}$ so that

$$u > \max\Big\{0, -\sqrt{2}T, -\tfrac{\sqrt{2}}{2}T - \tfrac{1}{2}R + \tfrac{1}{2}S + \tfrac{1}{4}, -\tfrac{\sqrt{2}}{2}T + \tfrac{1}{2}S\Big\}.$$

Then the following values are positive:

$$\begin{aligned} v &= u + \sqrt{2}T \\ w &= u + \tfrac{\sqrt{2}}{2}T + \tfrac{1}{2}R - \tfrac{1}{2}S - \tfrac{1}{4} \\ x &= u + \tfrac{\sqrt{2}}{2}T - \tfrac{1}{2}S. \end{aligned}$$

If we now set

$$A = \left(\frac{vw}{u}\right)^{\frac{1}{4}}, \quad B = \left(\frac{uw}{v}\right)^{\frac{1}{4}}, \quad C = \left(\frac{uv}{w}\right)^{\frac{1}{4}}, \quad D = \frac{x}{(uvw)^{\frac{1}{4}}},$$

then $A, B, C, D > 0$ and

$$\begin{aligned} \tfrac{1}{2} + 2A^2B^2 - 2ABCD &= R \\ B^2C^2 + A^2C^2 - 2ABCD &= S \\ \tfrac{\sqrt{2}}{2}(A^2 - B^2)C^2 &= T, \end{aligned}$$

as desired. □

8.1 Relationship to Lotay and Oliveira's Paper

In [18], the authors give examples of deformed G_2 instantons on tri-Sasakian 7-manifolds, and in particular on the Aloff-Wallach space X_1. In this section, I show that their deformed G_2 instantons on X_1 are a special case of the deformed G_2 instantons in Section 8. By comparing their equations to my own, I was able to find a minor mistake in their paper.

We have an embedding $f : U(2) \to \mathrm{SU}(3)$ given by

$$f(X) = \begin{pmatrix} X & 0 \\ 0 & (\det X)^{-1} \end{pmatrix}. \tag{36}$$

Note that SU(3) acts transitively on $\mathbb{CP}^2$, the space of complex lines in $\mathbb{C}^3$. Let $e_3 = (0,0,1) \in \mathbb{C}^3$, and let $L_3 \in \mathbb{CP}^2$ be the line generated by e_3. It is easy to see that everything in $\mathrm{im}(f)$ fixes L_3; specifically, $f(X)e_3 = (\det X)^{-1}e_3$. Conversely, if a matrix $A \in \mathrm{SU}(3)$ fixes L_3, then it must send e_3 to ze_3 for some $z \in U(1)$, which means that its third column is $(0,0,z)$. Then since the columns of A are orthogonal, the third row of A must be $(0,0,z)$, so $A = \mathrm{diag}(X,z)$ for some 2×2 matrix X. The fact that A is special unitary means that $X \in U(2)$ and $z = (\det X)^{-1}$, so $A = f(X)$. We conclude that the stabilizer of L_3 is $\mathrm{im}(f)$, and that there is an isomorphism $\mathrm{SU}(3)/\mathrm{im}(f) \cong \mathbb{CP}^2$, with projection $\sigma : \mathrm{SU}(3) \to \mathbb{CP}^2$ defined by $\sigma(A) = AL_3$. For any relatively prime[4] integers k,l,m such that $k+l+m=0$, the circle subgroup $T_{k,l,m}$ is contained in $\mathrm{im}(f)$, so the projection σ factors through $X_{k,l,m}$, giving a projection $\tau : X_{k,l,m} \to \mathbb{CP}^2$.

On $\mathbb{CP}^2$, we have the tautological complex line bundle $\mathcal{O}(-1) \to \mathbb{CP}^2$, with the fiber over $L \in \mathbb{CP}^2$ being L itself viewed as a line in $\mathbb{C}^3$. This corresponds to a principal $U(1)$-bundle $Q_{-1} \to \mathbb{CP}^2$ in which the fiber over $L \in \mathbb{CP}^2$ consists of the unit-length elements of L, with the $U(1)$ action given by scalar multiplication. We get a pullback bundle $\tau^*(Q_{-1}) \to X_{k,l,m}$, in which the fiber over $x \in X_{k,l,m}$ consists of the unit-length elements of $\tau(x)$. As a smooth manifold, $\tau^*(Q_{-1})$ is embedded in $X_{k,l,m} \times \mathbb{C}^3$.

We can parametrize $T_{k,l,m}$ by $\chi(z) = \mathrm{diag}(z^k, z^l, z^m)$ for $z \in U(1)$. Recall that for n an integer, the $U(1)$ bundle $P_n \to X_{k,l,m}$ can be defined as the quotient of $\mathrm{SU}(3) \times U(1)$ by the $U(1)$ action $(A,w)z = (A\chi(z), z^{-n}w)$. The projection from P_n to $X_{k,l,m}$ is simply the projection $\pi : \mathrm{SU}(3) \to X_{k,l,m}$ applied to the first component, and the $U(1)$ action on P_n is given by multiplication of the second component. We have the following proposition relating bundles over $X_{k,l,m}$.

Proposition 8.4. *Over the Aloff-Wallach space $X_{k,l,m}$, the pullback bundle $\tau^*(Q_{-1})$ is isomorphic to P_m, with isomorphism $g : P_m \to \tau^*(Q_{-1})$ given by $g(A,w) = (\pi(A), wAe_3)$. Here Ae_3 can be thought of as the third column of A.*

Proof. To see that g is a well-defined function on P_m, note that

$$\begin{aligned} g(A\chi(z), z^{-n}w) &= (\pi(A\chi(z)), z^{-n}wA\chi(z)e_3) \\ &= (\pi(A), z^{-n}wz^nAe_3) = g(A,w). \end{aligned}$$

[4] This condition excludes $0,0,0$ but not $1,0,-1$ or its permutations.

And $g(A, w)$ is in fact contained in $\tau^*(Q_{-1})$, since wAe_3 is a unit vector in $\tau(\pi(A)) = \sigma(A) = AL_3$. Lastly, it is easy to see that g commutes with the projections to $X_{k,l,m}$ as well as the $U(1)$ actions. $\square$

Along with the tautological line bundle $\mathcal{O}(-1) \to \mathbb{CP}^2$, we have a family of Hermitian line bundles $\mathcal{O}(n)$ for $n \in \mathbb{Z}$, defined as the tensor product of $\mathcal{O}(-1)$ with itself $-n$ times. (If n is positive, this means taking the dual and then tensoring with itself n times.) I will denote the principal $U(1)$ bundle corresponding to $\mathcal{O}(n)$ as Q_n, generalizing the notation Q_{-1}. The operation of tensoring a Hermitian line bundle with itself n times corresponds to taking the λ_n associated bundle of a principal $U(1)$ bundle, where $\lambda_n : U(1) \to U(1)$ is the homomorphism $\lambda_n(z) = z^n$. (It also corresponds to multiplying the Chern class by n.) This means that Q_n is the λ_{-n} associated bundle of Q_{-1}. So we have the following proposition, generalizing Proposition 8.4:

Proposition 8.5. *For all $n \in \mathbb{Z}$, the bundle $\tau^*(Q_n)$ over $X_{k,l,m}$ is isomorphic to $P_{-nm} = P_{n(k+l)}$.*

Proof. Note that $\tau^*(Q_n)$ is the λ_{-n} associated bundle of $\tau^*(Q_{-1})$, while the λ_{-n} associated bundle of P_m is P_{-nm} (since $\lambda_{-n} \circ \lambda_m = \lambda_{-nm}$). $\square$

Note that the projection $\sigma : \mathrm{SU}(3) \to \mathbb{CP}^2$ factors as

$$\sigma : \mathrm{SU}(3) \xrightarrow{\tilde{\sigma}} S^5 \xrightarrow{h} \mathbb{CP}^2,$$

where $\tilde{\sigma}(A) = Ae_3$, and h is the Hopf fibration taking a unit vector in $\mathbb{C}^3$ to the complex line containing it. (Unlike σ, the map $\tilde{\sigma}$ does not necessarily factor through $X_{k,l,m}$, because matrices representing the same element of $X_{k,l,m}$ can have different third columns.) I will take the Riemannian metric on $\mathbb{CP}^2$ to be the one making h into a Riemannian submersion, without any normalization factor. This gives a Kähler structure on $\mathbb{CP}^2$ with Kähler form $\omega \in \Omega^2(\mathbb{CP}^2)$. We will investigate the pullback $\tau^*\omega$ on $X_{k,l,m}$ by investigating the pullback $\sigma^*\omega = \tilde{\sigma}^* h^* \omega$ on $\mathrm{SU}(3)$.

Recall the basis $Y, Z_1, \ldots, Z_7$ for $\mathfrak{su}(3) = T_1\mathrm{SU}(3)$ from Section 6. By taking the third column of each matrix, we see that

$$\begin{aligned}
d\tilde{\sigma}(Y) &= \left(0, 0, \frac{im}{s}\right) & d\tilde{\sigma}(Z_1) &= \left(0, 0, \frac{i(k-l)}{s\sqrt{3}}\right) \\
d\tilde{\sigma}(Z_2) &= \left(0, \frac{-1}{\sqrt{2}}, 0\right) & d\tilde{\sigma}(Z_3) &= \left(0, \frac{i}{\sqrt{2}}, 0\right) \\
d\tilde{\sigma}(Z_4) &= \left(\frac{1}{\sqrt{2}}, 0, 0\right) & d\tilde{\sigma}(Z_5) &= \left(\frac{i}{\sqrt{2}}, 0, 0\right) \\
d\tilde{\sigma}(Z_6) &= (0, 0, 0) & d\tilde{\sigma}(Z_7) &= (0, 0, 0);
\end{aligned}$$

these are elements of $T_{e_3}S^5 \subseteq \mathbb{C}^3$. Note that the kernel of h in $T_{e_3}S^5$ consists of real multiples of ie_3, so $d\sigma(Y) = d\sigma(Z_1) = 0$. The orthogonal complement of this kernel — a 2-dimensional complex subspace of $\mathbb{C}^3$ — is mapped bijectively by h onto $T_{L_3}\mathbb{CP}^2$, with $h^*\omega$ acting as the restriction of the standard Kähler form on $\mathbb{C}^3$. Thus

$$h^*\omega(d\tilde{\sigma}(Z_2), d\tilde{\sigma}(Z_3)) = \frac{-1}{2}, \qquad h^*\omega(d\tilde{\sigma}(Z_4), d\tilde{\sigma}(Z_5)) = \frac{1}{2},$$

with all other combinations 0. We conclude that, on $T_1\mathrm{SU}(3)$,

$$\sigma^*\omega = \tilde{\sigma}^*h^*\omega = \frac{-1}{2}\alpha_2 \wedge \alpha_3 + \frac{1}{2}\alpha_4 \wedge \alpha_5. \tag{37}$$

Note that SU(3) acts on the left on both itself and $\mathbb{CP}^2$, and that σ commutes with these actions. Specifically, for $A, B \in \mathrm{SU}(3)$, we have $\sigma(AB) = ABL_3 = A\sigma(B)$. The action of SU(3) on $\mathbb{CP}^2$ preserves both the complex structure and the Riemannian metric, so ω is SU(3)-invariant. This implies that $\sigma^*\omega$ is also (left) SU(3) invariant, so the formula (37) for $\sigma^*\omega$ applies not just at $1 \in \mathrm{SU}(3)$ but on all of SU(3).

Since $\sigma = \tau \circ \pi$, we have $\pi^*\tau^*\omega = -\frac{1}{2}\alpha_{23} + \frac{1}{2}\alpha_{45} \in \Omega^2(\mathrm{SU}(3))$. Now since π^* is injective on forms, there is only one possibility for $\tau^*\omega \in \Omega^2(X_{k,l,m})$, namely $\tau^*\omega = -\frac{1}{2}\alpha_{23} + \frac{1}{2}\alpha_{45}$, where as in previous sections I am using the same notation for invariant forms on $X_{k,l,m}$ and their pullbacks on SU(3).

For remainder of this section, we will restrict to the case of $X_1 = X_{1,1,-2}$. Lotay and Oliveira study the bundles $\tau^*\mathcal{O}(n)$ over X_1 (using my notation), and obtain their base connection on $\tau^*\mathcal{O}(n)$ by pulling back a Hermitian connection on $\mathcal{O}(n)$ with harmonic curvature. Of course, we can equivalently take the $U(1)$ bundle $\tau^*Q_n \to X_1$ and pull back a connection on Q_n with harmonic curvature.

Proposition 8.6. *Let n be an integer, and let A be a connection on $Q_n \to \mathbb{CP}^2$ with harmonic curvature. Then the connection τ^*A on $\tau^*Q_n \to X_1$ is related by a bundle isomorphism to the connection A_{2n} on $P_{2n} \to X_1$.*

Proof. Since ω is harmonic and $b^2(\mathbb{CP}^2) = 1$, every harmonic 2-form on $\mathbb{CP}^2$ is a scalar multiple of ω. Thus $F_A = i\lambda\omega$ for some $\lambda \in \mathbb{R}$. This implies that

$$F_{\tau^*A} = i\lambda\tau^*\omega = -\frac{i\lambda}{2}(\alpha_{23} - \alpha_{45}).$$

From equation (26) in a previous section we know that the curvature of A_{2n} on $P_{2n} \to X_1$ is

$$F_{2n} = in(\alpha_{23} - \alpha_{45})$$

Since τ^*Q_n is isomorphic to P_{2n} by Proposition 8.5, we know that F_{τ^*A} and F_{2n} have the same cohomology class. Since they are both scalar multiples of $\alpha_{23} - \alpha_{45}$, they must actually be equal, with $\lambda = -2n$. Using some bundle isomorphism between τ^*Q_n and P_{2n}, we can carry over τ^*A to a connection A' on P_{2n} with the same curvature. Note that $A' - A_{2n}$ is a closed 1-form on X_1; since $b^1(X_1) = 0$, we find that $A' - A_{2n}$ is exact, and thus A' and A_{2n} are related by a gauge transformation (or bundle automorphism). $\square$

Recall that the map $\sigma : \mathrm{SU}(3) \to \mathbb{CP}^2$ can thought of as quotienting by the right action of $U(2)$ defined by the homomorphism $f : U(2) \to \mathrm{SU}(3)$ as in (36). Note that T_1 is the image under f of the center of $U(2)$, so everything in T_1 commutes with everything in $\operatorname{im} f$. As a result, the right action of $U(2)$ on $\mathrm{SU}(3)$ descends to a right action of $U(2)$ on X_1; we can think of the map $\tau : X_1 \to \mathbb{CP}^2$ as quotienting by this $U(2)$ action.

The action of $U(2)$ on X_1 is not free: indeed, the stabilizer of every point is the center of $U(2)$, so we can think of it as a free action of the projective unitary group $\mathrm{PU}(2) \cong \mathrm{SO}(3)$. Alternatively, we can think of it as a locally free action of $\mathrm{SU}(2)$, which is a double cover of $\mathrm{PU}(2)$. The Lie algebra $\mathfrak{su}(2)$ of $\mathrm{SU}(2)$ has the following basis:

$$Y_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad Y_2 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \qquad Y_3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

These satisfy the commutation relations $[Y_1, Y_2] = 2Y_3$ and cyclic permutations thereof. Under the Lie algebra homomorphism $df : \mathfrak{u}(2) \to \mathfrak{su}(3)$, we have

$$df(Y_1) = -\sqrt{2}Z_6 \qquad df(Y_2) = -\sqrt{2}Z_7 \qquad df(Y_3) = -\sqrt{2}Z_1$$

which means that the left translations of $-\sqrt{2}Z_6, -\sqrt{2}Z_7, -\sqrt{2}Z_1$ are the generating vector fields for the right action of $\mathrm{SU}(2)$ on $\mathrm{SU}(3)$. Recalling that the left translation of $Y = \frac{1}{\sqrt{6}} \operatorname{diag}(i, i, -2i)$ generates the action of T_1 on $\mathrm{SU}(3)$, and noting that $[Y, Z_6] = [Y, Z_7] = [Y, Z_1] = 0$, we see that these generating vector fields are T_1-invariant and thus descend to vector fields ξ_1, ξ_2, ξ_3 on X_1, which are the generating vector fields for the action of $\mathrm{SU}(2)$ on X_1. Along with the metric

$$g = \frac{1}{2}\left(\alpha_1^2 + \alpha_6^2 + \alpha_7^2\right) + \frac{1}{4}\left(\alpha_2^2 + \alpha_3^2 + \alpha_4^2 + \alpha_5^2\right),$$

which makes them into unit vectors, ξ_1, ξ_2, ξ_3 determine a tri-Sasakian structure on X_1. The 1-forms related to ξ_1, ξ_2, ξ_3 by g are

$$\eta_1 = \frac{-1}{\sqrt{2}}\alpha_6 \qquad \eta_2 = \frac{-1}{\sqrt{2}}\alpha_7 \qquad \eta_3 = \frac{-1}{\sqrt{2}}\alpha_1. \tag{38}$$

Let $\overline{\eta} = \sum_i \eta_i \otimes Y_i$ be an $\mathfrak{su}(2)$-valued 1-form on X_1. Following [18, Section 2], we define $\omega_1, \omega_2, \omega_3$ by

$$d\overline{\eta} + \frac{1}{2}[\overline{\eta} \wedge \overline{\eta}] = \sum_i (-2\omega_i) \otimes T_i.$$

We compute

$$\begin{aligned}
\omega_1 &= \frac{1}{2\sqrt{2}} d\alpha_6 - \frac{1}{2}\alpha_7 \wedge \alpha_1 = -\frac{1}{4}\alpha_2 \wedge \alpha_4 + \frac{1}{4}\alpha_3 \wedge \alpha_5 \\
\omega_2 &= \frac{1}{2\sqrt{2}} d\alpha_7 - \frac{1}{2}\alpha_1 \wedge \alpha_6 = \frac{1}{4}\alpha_2 \wedge \alpha_5 + \frac{1}{4}\alpha_3 \wedge \alpha_4 \\
\omega_3 &= \frac{1}{2\sqrt{2}} d\alpha_1 - \frac{1}{2}\alpha_6 \wedge \alpha_7 = -\frac{1}{4}\alpha_2 \wedge \alpha_3 - \frac{1}{4}\alpha_4 \wedge \alpha_5.
\end{aligned}$$

Lotay and Oliveira define their G_2 structures (on an arbitrary tri-Sasakian 7-manifold) as

$$\varphi_{t,\varepsilon} = \varepsilon t^3 \eta_{123} - t(\eta_1 \wedge \omega_1 + \eta_2 \wedge \omega_2 + \varepsilon \eta_3 \wedge \omega_3),$$

where $t \in \mathbb{R}_{>0}$ and $\varepsilon \in \{\pm 1\}$. In the tri-Sasakian structure on X_1 that we are currently considering, this gives

$$\varphi_{t,\varepsilon} = -\varepsilon \frac{\sqrt{2}}{8} t\, \alpha_1 \wedge (\alpha_{23} + \alpha_{45} + 2t^2 \alpha_{67}) - \frac{\sqrt{2}}{8} t(\alpha_{246} - \alpha_{257} - \alpha_{347} - \alpha_{356}),$$

which is equal to the G_2 structure defined in (21) with

$$A = B = \frac{1}{2}, \quad C = -\frac{\sqrt{2}}{2} t, \quad D = -\varepsilon \frac{\sqrt{2}}{2} t. \tag{39}$$

Two of these G_2 structures are nearly parallel: $\varphi_{1,-1}$ satisfies (25) with $\lambda = 4$, and $\varphi_{1/\sqrt{5},+1}$ satisfies (24) with $\lambda = 12/\sqrt{5}$. Lotay and Oliveira call the former φ^{ts} (since its metric is the tri-Sasakian metric) and the latter φ^{np}.

They consider connections on $\tau^* Q_k$ (my notation; here $k \in \mathbb{Z}$ is arbitrary) of the form $\tau^* A_0 + i(a_1\eta_1 + a_2\eta_2 + a_3\eta_3)$, where A_0 has harmonic curvature and $a_1, a_2, a_3 \in \mathbb{R}$. By Proposition 8.6 and the definition of η_1, η_2, η_3 in (38), this corresponds to the connection $A_{2k,\vec{c}}$ on $P_{2k} \to X_1$ where

$$c_1 = \frac{-a_3}{\sqrt{2}}, \quad c_6 = \frac{-a_1}{\sqrt{2}}, \quad c_7 = \frac{-a_2}{\sqrt{2}}. \tag{40}$$

If we apply the substitutions in (39) and (40), as well as $n = 2k$, to the deformed G_2 instanton equations (30), (31), (32), we obtain the following deformed G_2 instanton conditions for Lotay and Oliveira's connections on P_{2k}:

$$\begin{aligned}
\left(\tfrac{1}{2}\varepsilon t^2 - \tfrac{1}{4} - 4k^2 + a_1^2 + a_2^2 + a_3^2\right) a_1 &= 0 \\
\left(\tfrac{1}{2}\varepsilon t^2 - \tfrac{1}{4} - 4k^2 + a_1^2 + a_2^2 + a_3^2\right) a_2 &= 0 \\
\left(\tfrac{1}{2} t^2 - \tfrac{1}{4} - 4k^2 + a_1^2 + a_2^2 + a_3^2\right) a_3 &= 0.
\end{aligned}$$

There is a discrepancy between these equations and the corresponding equations in the original published version of [18, Section 4.3]. Specifically, Lotay and Oliveira had $\frac{1}{4}k^2$ instead of $4k^2$. This was due to the simultaneous use of two differently scaled Fubini-Study metrics on $\mathbb{CP}^2$. I have pointed this out to them and they plan on making corrections.

A Invariant Forms on Aloff-Wallach Spaces

For $q \in \mathbb{Q} \cap [0,1]$ and $p \in \mathbb{Z} \cap [0,7]$, I will denote by $\Omega^p_{\mathrm{inv}}(X_q, \mathbb{R})$ the real vector space of SU(3)-invariant real p-forms on X_q. Likewise, I will denote by $\Omega^p_{\mathrm{inv}}(X_q, \mathbb{C})$ the complex vector space of SU(3)-invariant complex p-forms on X_q. Since a complex p-form is SU(3)-invariant if and only if both its real and its imaginary part are SU(3)-invariant, we see that $\Omega^p_{\mathrm{inv}}(X_q, \mathbb{C})$ is the complexification of $\Omega^p_{\mathrm{inv}}(X_q, \mathbb{R})$. In particular, the complex dimension of $\Omega^p_{\mathrm{inv}}(X_q, \mathbb{C})$ is equal to the real dimension of $\Omega^p_{\mathrm{inv}}(X_q, \mathbb{R})$; I will denote this dimension as $D(p;q)$. Note that $D(p;q)$ is indeed finite, since the action of SU(3) on X_q is transitive.

Elements of $\Omega^p_{\mathrm{inv}}(X_q, \mathbb{C})$ correspond (via pullback) to complex p-forms on SU(3) which are (i) left-invariant, (ii) horizontal with respect to the projection $\mathrm{SU}(3) \to X_q$, and (iii) invariant under the right action of T_q. The space of p-forms on SU(3) satisfying (i) and (ii) is the pth exterior power of the complex vector space spanned by $\alpha_1, \ldots, \alpha_7$. Since we are using complex coefficients, we can instead use the basis $\alpha_1, \beta_1, \overline{\beta_1}, \beta_2, \overline{\beta_2}, \beta_3, \overline{\beta_3}$ (where as in Section 6, we have $\beta_j = \alpha_{2j} + i\alpha_{2j+1}$). Recall that the matrix $A(t) = \mathrm{diag}(e^{ikt}, e^{ilt}, e^{imt}) \in T_q$ acts on these basis elements by

$$\begin{aligned} &R^*_{A(t)}\alpha_1 = \alpha_1, && R^*_{A(t)}\beta_1 = e^{i(l-m)t}\beta_1, && R^*_{A(t)}\overline{\beta_1} = e^{i(m-l)t}\overline{\beta_1}, \\ & && R^*_{A(t)}\beta_2 = e^{i(m-k)t}\beta_2, && R^*_{A(t)}\overline{\beta_2} = e^{i(k-m)t}\overline{\beta_2}, \\ & && R^*_{A(t)}\beta_3 = e^{i(k-l)t}\beta_3, && R^*_{A(t)}\overline{\beta_3} = e^{i(l-k)t}\overline{\beta_3}. \end{aligned}$$

In general, $A(t)$ acts on any wedge product of these basis elements by scaling it by some complex number. Thus an element of the pth exterior power is invariant under the action of T_q if and only if every basis wedge for which it has a nonzero coefficient is invariant under the action of T_q. We conclude that $D(p;q)$ is equal to the number of p-wedges of $\alpha_1, \beta_1, \overline{\beta_1}, \beta_2, \overline{\beta_2}, \beta_3, \overline{\beta_3}$ on which T_q acts trivially; this is the number of ways of writing 0 as a sum of p of the following integers (with no repeats and ignoring order):

$$\begin{array}{lll} 0 & l-m & m-l \\ & m-k & k-m \\ & k-l & l-k. \end{array} \tag{41}$$

The sum of all of these integers is 0, so by taking complements, we have $D(p;q) = D(7-p;q)$. Clearly $D(0;q) = D(7;q) = 1$ (corresponding to the fact that a scalar function on X_q is SU(3)-invariant if and only if it is constant). If $q = 1$, then $k = l$ and we have $D(1;q) = 3$; otherwise, k, l, m are all distinct and we have $D(1;q) = 1$.

To compute $D(2;q)$ and $D(3;q)$, we first suppose that $q \notin \{0,1\}$. Then $k > l > 0$, so (recalling that $m = -k-l$), we have $0 < k-l < l-m < k-m$. So to obtain 0 from two of the integers in (41), we cannot use the 0 and must add two on the same row, giving three possibilities. Thus $D(2;q) = 3$ for $q \notin \{0,1\}$. Likewise, there are three ways to obtain 0 from three of the integers in (41) if one of them is 0. If we do not make use of the 0, it is pretty easy to see that the only triples that sum to 0 are the second and third columns. We conclude that $D(3;q) = 5$ for $q \notin \{0,1\}$.

If $q = 0$, the seven numbers are

$$0 \quad +1 \quad -1 \quad +1 \quad -1 \quad +2 \quad -2.$$

There are five pairs here that add to 0: four of the form $+1, -1$ and one of the form $+2, -2$. And there are seven triples that add to 0: the aforementioned pairs with the addition of 0, along with two triples of the form $\pm 1, \pm 1, \mp 2$. So $D(2;0) = 5$ and $D(3;0) = 7$.

Lastly, if $q = 1$, the seven numbers are

$$0 \quad 0 \quad 0 \quad +3 \quad -3 \quad +3 \quad -3.$$

There are seven pairs here that add to 0: three consisting of two 0s and four of the form $+3, -3$. And there are thirteen triples that add to 0: one consisting of three 0s and twelve of the form $0, +3, -3$. We conclude that $D(2;1) = 7$ and $D(3;1) = 13$.

These results are summarized in the following.

Proposition A.1. *If $q \in \mathbb{Q} \cap [0,1]$ and $p \in \mathbb{Z} \cap [0,7]$, then the dimension $D(p;q)$ of the vector space of* SU(3)*-invariant p-forms on X_q is given by the following table:*

$p =$	$0,7$	$1,6$	$2,5$	$3,4$
$q \notin \{0,1\}$	1	1	3	5
$q = 0$	1	1	5	7
$q = 1$	1	3	7	13

Note that the method of proof gives us an explicit basis for the vector space $\Omega^p_{\mathrm{inv}}(X_q, \mathbb{C})$, namely those p-wedges of $\alpha_1, \beta_1, \overline{\beta_1}, \beta_2, \overline{\beta_2}, \beta_3, \overline{\beta_3}$ corresponding to sets of integers in (41) that sum to 0. These basis forms are either real or come in conjugate pairs; by replacing the conjugate pairs with their real and imaginary parts, we obtain a basis consisting only of real forms, hence a basis for $\Omega^p_{\mathrm{inv}}(X_q, \mathbb{R})$.

Before continuing, I will make some remarks on the nature of G_2 structures, citing [4]. If V is a 7-dimensional real vector space, then the G_2 structures on V constitute an *open* subset of $\Lambda^3 V^*$, which is often denoted $\Lambda^3_+ V^*$. The set $\Lambda^3_+ V^*$ has two connected components, which are negatives of each other; G_2 structures in different connected components determine opposite orientations. We have a map $S : \Lambda^3_+ V^* \to \Lambda^4 V^*$ taking a G_2 structure to its coassociative 4-form. One can show that S is a double covering onto a connected open set $\Lambda^4_+ V^* \subseteq \Lambda^4 V^*$, with $S(-\varphi) = S(\varphi)$ for all $\varphi \in \Lambda^3_+ V^*$. This means that, instead of defining a G_2 structure by an element of $\Lambda^3_+ V^*$, we can equivalently define it by an element of $\Lambda^4_+ V^*$ along with an orientation on V. Likewise, if M is a 7-dimensional manifold, we can define a G_2 structure on M by a section of $\Lambda^4_+ T^* M$ along with an orientation on M.

An invariant form on X_q is determined by its value at a single point. So if we choose some point $x \in X_q$, we can identify $\Omega^3_{\text{inv}}(X_q, \mathbb{R})$ with a vector subspace $W \subseteq \Lambda^3 T^*_x X_q$ of dimension 5, 7, or 13 as given in Proposition A.1. Then the invariant G_2 structures on X_q are identified with $W \cap \Lambda^3_+ T^*_x X_q$. We know that this set is nonempty, since X_q does admit invariant G_2 structures (e.g. the ones in (21)). So by openness of $\Lambda^3_+ T^*_x X_q$, it has the same dimension as W. That is,

Proposition A.2. *The space of* $\mathrm{SU}(3)$*-invariant* G_2 *structures on* X_q *has dimension*

- *5 if* $q \notin \{0, 1\}$*,*
- *7 if* $q = 0$*,*
- *13 if* $q = 1$*.*

In [6], the authors parametrize the full 5-dimensional space of invariant G_2 structures in the case $q \notin \{0, 1\}$; they find that the coclosed G_2 structures constitute a 4-dimensional subspace, which we have seen in (21). I now show that, although X_0 has more invariant G_2 structures than X_q with $q \notin \{0, 1\}$, it does not have more invariant *coclosed* G_2 structures.

Proposition A.3. *On* X_0*, every* $\mathrm{SU}(3)$*-invariant coclosed* G_2 *structure is of the form* (21)*.*

Proof. The space $\Omega^4_{\text{inv}}(X_0, \mathbb{R})$, which by Proposition A.1 is 7-dimensional, has a basis given by

$$
\begin{aligned}
-\tfrac{1}{4}\beta_2 \wedge \overline{\beta_2} \wedge \beta_3 \wedge \overline{\beta_3} &= \alpha_{4567} \\
-\tfrac{1}{4}\beta_1 \wedge \overline{\beta_1} \wedge \beta_3 \wedge \overline{\beta_3} &= \alpha_{2367} \\
-\tfrac{1}{4}\beta_1 \wedge \overline{\beta_1} \wedge \beta_2 \wedge \overline{\beta_2} &= \alpha_{2345} \\
-\tfrac{1}{2}\mathrm{Re}(\beta_1 \wedge \overline{\beta_3} \wedge \beta_2 \wedge \overline{\beta_2}) &= \alpha_{2457} - \alpha_{3456} \\
-\tfrac{1}{2}\mathrm{Im}(\beta_1 \wedge \overline{\beta_3} \wedge \beta_2 \wedge \overline{\beta_2}) &= \alpha_{2456} + \alpha_{3457} \\
\mathrm{Re}(\alpha_1 \wedge \beta_1 \wedge \beta_2 \wedge \beta_3) &= \alpha_{1246} - \alpha_{1257} - \alpha_{1347} - \alpha_{1356} \\
\mathrm{Im}(\alpha_1 \wedge \beta_1 \wedge \beta_2 \wedge \beta_3) &= \alpha_{1247} + \alpha_{1256} + \alpha_{1346} - \alpha_{1357}
\end{aligned}
$$

Denote these respectively as $\gamma_1, \ldots, \gamma_7$. An arbitrary invariant 4-form on X_0 is given by $\chi = u_1\gamma_1 + \cdots + u_7\gamma_7$ with $u_1, \ldots, u_7 \in \mathbb{R}$. If we compute $d\chi$ using the formulas for $d\alpha_1, \ldots, d\alpha_7$ in (23), we see that $d\chi = 0$ if and only if $u_4 = u_5 = u_6 = 0$. Thus the closed invariant 4-forms on X_0 are given by

$$
\begin{aligned}
&u_1\gamma_1 + u_2\gamma_2 + u_3\gamma_3 + u_7\gamma_7 \\
&= u_1\alpha_{4567} + u_2\alpha_{2367} + u_3\alpha_{2345} + u_7(\alpha_{1247} + \alpha_{1256} + \alpha_{1346} - \alpha_{1357}).
\end{aligned}
$$

with $u_1, u_2, u_3, u_7 \in \mathbb{R}$. These are well-defined and closed not only on X_0, but on every Aloff-Wallach space. Those with $u_1, u_2, u_3 > 0$ and $u_7 \neq 0$ are the 4-forms seen in (22), and hence, when supplemented by an orientation, determine one of the G_2 structures in (21). The rest cannot determine G_2 structures: if one did, we could use the same formula to get an invariant coclosed G_2 structure on (e.g.) $X_{1/2}$ not of the form in (21), contradicting the results of [6]. $\square$

However, X_1 does have more invariant coclosed G_2 structures than the other Aloff-Wallach spaces, in fact a 10-dimensional space of them, as I show in the following.

Proposition A.4. *On X_1, the space of $\mathrm{SU}(3)$-invariant coclosed G_2 structures is 10-dimensional.*

Proof. By Proposition A.1, we know that $\Omega^4_{\mathrm{inv}}(X_1, \mathbb{R})$ is 13-dimensional. We have a basis for the complexification $\Omega^4_{\mathrm{inv}}(X_1, \mathbb{C})$ consisting of the 4-form $\beta_1 \wedge \overline{\beta_1} \wedge \beta_2 \wedge \overline{\beta_2}$ along with all 12 possible wedges of one of $\{\beta_1, \overline{\beta_2}\}$, one of $\{\overline{\beta_1}, \beta_2\}$, and two of $\{\alpha_1, \beta_3, \overline{\beta_3}\}$. Three of these basis forms are real, and the remaining ten come in conjugate pairs. By replacing the conjugate pairs with their real and imaginary parts (and then applying some further modifications), we can obtain the following basis for $\Omega^4_{\mathrm{inv}}(X_1, \mathbb{R})$:

$$\begin{aligned}
-\tfrac{1}{4}\beta_2 \wedge \overline{\beta_2} \wedge \beta_3 \wedge \overline{\beta_3} &= \alpha_{4567} \\
-\tfrac{1}{4}\beta_1 \wedge \overline{\beta_1} \wedge \beta_3 \wedge \overline{\beta_3} &= \alpha_{2367} \\
-\tfrac{1}{4}\beta_1 \wedge \overline{\beta_1} \wedge \beta_2 \wedge \overline{\beta_2} &= \alpha_{2345} \\
\tfrac{1}{2}\mathrm{Re}(\beta_1 \wedge \beta_2 \wedge \beta_3 \wedge \overline{\beta_3}) &= \alpha_{2567} + \alpha_{3467} \\
-\tfrac{1}{2}\mathrm{Im}(\beta_1 \wedge \beta_2 \wedge \beta_3 \wedge \overline{\beta_3}) &= \alpha_{2467} - \alpha_{3567} \\
\tfrac{1}{2}\mathrm{Re}(\alpha_1 \wedge \beta_1 \wedge \overline{\beta_1} \wedge \beta_3) &= \alpha_{1237} \\
-\tfrac{1}{2}\mathrm{Im}(\alpha_1 \wedge \beta_1 \wedge \overline{\beta_1} \wedge \beta_3) &= \alpha_{1236} \\
\tfrac{1}{2}\mathrm{Re}(\alpha_1 \wedge \beta_2 \wedge \overline{\beta_2} \wedge \beta_3) &= \alpha_{1457} \\
-\tfrac{1}{2}\mathrm{Im}(\alpha_1 \wedge \beta_2 \wedge \overline{\beta_2} \wedge \beta_3) &= \alpha_{1456} \\
\tfrac{1}{2}\mathrm{Re}(\alpha_1 \wedge \beta_1 \wedge \beta_2 \wedge \beta_3 + \alpha_1 \wedge \overline{\beta_1} \wedge \overline{\beta_2} \wedge \beta_3) &= \alpha_{1246} - \alpha_{1356} \\
-\tfrac{1}{2}\mathrm{Re}(\alpha_1 \wedge \beta_1 \wedge \beta_2 \wedge \beta_3 - \alpha_1 \wedge \overline{\beta_1} \wedge \overline{\beta_2} \wedge \beta_3) &= \alpha_{1257} + \alpha_{1347} \\
\tfrac{1}{2}\mathrm{Im}(\alpha_1 \wedge \beta_1 \wedge \beta_2 \wedge \beta_3 + \alpha_1 \wedge \overline{\beta_1} \wedge \overline{\beta_2} \wedge \beta_3) &= \alpha_{1247} - \alpha_{1357} \\
\tfrac{1}{2}\mathrm{Im}(\alpha_1 \wedge \beta_1 \wedge \beta_2 \wedge \beta_3 - \alpha_1 \wedge \overline{\beta_1} \wedge \overline{\beta_2} \wedge \beta_3) &= \alpha_{1256} + \alpha_{1346}
\end{aligned}$$

Call these $\gamma_1, \dots, \gamma_{13}$. An invariant real 4-form on X_1 can be written as $\chi = u_1\gamma_1 + \cdots + u_{13}\gamma_{13}$ with $u_1, \dots, u_{13} \in \mathbb{R}$. Taking the exterior derivative, we see that $d\chi = 0$ if and only if

$$u_{10} - u_{11} = 0, \quad u_4 + \tfrac{1}{2}u_7 + \tfrac{1}{2}u_9 = 0, \quad u_5 + \tfrac{1}{2}u_6 + \tfrac{1}{2}u_8 = 0.$$

Thus the invariant closed 4-forms on X_1 form a 10-dimensional vector space, which (choosing some point $x \in X_1$) can be identified with a 10-dimensional vector subspace $W \subseteq \Lambda^4 T^*_x X_1$. The intersection $J = W \cap \Lambda^4_+ T^*_x X_1$ can be thought of as the space of invariant coclosed G_2 structures on X_1 modulo orientation. J is nonempty, since X_1 has some invariant coclosed G_2 structures (e.g. the ones in (21)). So J is 10-dimensional by the openness of $\Lambda^4_+ T^*_x X_1$. $\square$

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