1 Morphism preservation

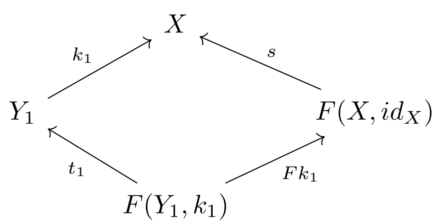
Suppose we have the following:

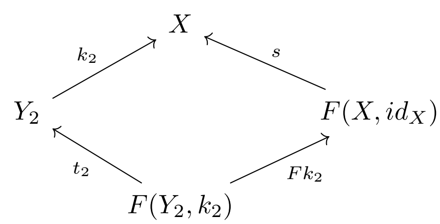
- a category C
- an object $X \in C$
- a functor $F : C/X \to C$
- a morphism $s : F(X, id_X) \to X$

I will say that an object $(Y, k) \in C/X$ *preserves* s if there is a morphism $t : F(Y, k) \to Y$ such that $kt = sFk$.

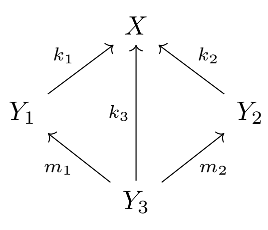
2 Proof: if two objects preserve a morphism, so does their product

Assume there are two objects $(Y_1, k_1), (Y_2, k_2) \in C/X$ which both preserve s. In other words, the following diagrams commute for some t_1 and t_2:

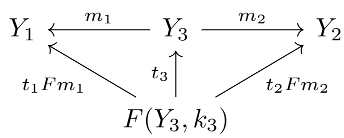




Let (Y_3, k_3) be the product $(Y_1, k_1) \times (Y_2, k_2)$, or equivalently, the pullback of k_1 and k_2 in C.



Note that $k_1 t_1 F m_1 = sFk_1 F m_1 = sFk_2 F m_2 = k_2 t_2 F m_2$. This means that the object $F(Y_3, k_3)$ along with the morphisms $t_1 F m_1$ and $t_2 F m_2$ forms a cone over k_1 and k_2. Due to the defining property of the pullback, there must be a unique morphism $t_3 : F(Y_3, k_3) \to Y_3$ such that the following diagram commutes:



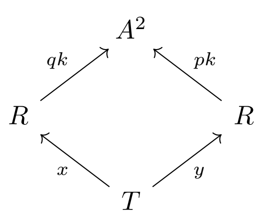
We can now show that (Y_3, k_3) preserves s: $k_3t_3 = k_1m_1t_3 = k_1t_1Fm_1 = sFk_1Fm_1 = sFk_3$.

3 Properties of relations on objects

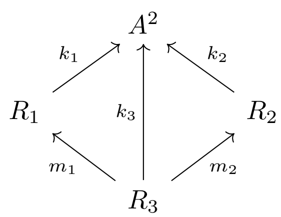
One application of the preceding theorem concerns relations on objects. A relation on a set A can be defined as a subset $R \subseteq A^2$. More generally, in any category C, a relation on an object A is another object R along with a monomorphism $k : R \hookrightarrow A^2$ (where A^2 denotes the product of A with itself). Equivalently, a relation on an object A is an object (R, k) in the overcategory C/A^2 where k is monic.

Using this definition, we can define some properties that relations might have.

- (R, k) is *reflexive* if there is a morphism $t : A \to R$ such that $kt = \langle id_A, id_A \rangle$.
- (R, k) is *symmetric* if there is a morphism $t : R \to R$ such that $kt = \langle q, p \rangle k$. Here, p and q are the projections from A^2 to A, and therefore $\langle q, p \rangle : A^2 \to A^2$ can be thought of as a function that switches the items in a pair.
- (R, k) is *transitive* if there is a morphism $t : T \to R$ such that $kt = \langle pkx, qky \rangle$, where T, x, and y are defined by the pullback diagram



We can define the intersection of two relations (R_1, k_1) and (R_2, k_2) on the same object A as the pullback of $k1$ and $k2$.



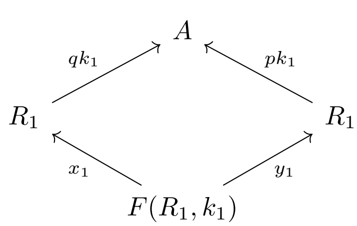
Because pullbacks preserve monomorphisms, all of the morphisms in the diagram above are monic, and (R_3, k_3) is a relation on A.

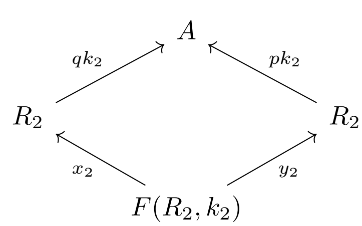
4 Proof: the intersection of two relations inherits their properties

An interesting question to ask is whether the properties of R_1 and R_2 are inherited by R_3. To answer this question, all we need to do is define some functor $F : C/A^2 \to C$ and some morphism $s : F(A^2, id_{A^2}) \to A^2$ such that relations with a certain property (e.g. reflexivity) are exactly those which preserve s. By the theorem I proved in section 2, it is then guaranteed that if two relations have that property, their intersection will also have that property.

- Let F be the constant functor at A, and $s : A \to A^2$ the diagonal morphism $\langle id_A, id_A \rangle$. Then a relation is reflexive if and only if it preserves s.
- Let F be the forgetful functor from C/A^2 to C, and $s : A^2 \to A^2$ the "swap function" $\langle q, p \rangle$. Then a relation is symmetric if and only if it preserves s.
- The case for transitive relations is more complicated. We define F to be a functor that assigns each object $(R, k) \in C/A^2$ to the pullback of qk and pk.

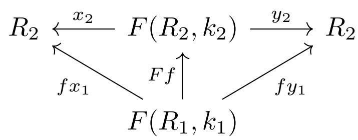
  We also need to define how F maps morphisms. Suppose there is a morphism from some object $(R_1, k_1) \in C/A^2$ to another object (R_2, k_2). By definition, this corresponds to a morphism $f : R_1 \to R_2$ such that $k_2 f = k_1$. Let $x_1, y_1 : F(R_1, k_1) \to R_1$ and $x_2, y_2 : F(R_2, k_2) \to R_2$ be pullback projections.

  Note that $qk_2 f x_1 = qk_1 x_1 = pk_1 y_1 = pk_2 f y_1$. This means that $F(R_1, k_1)$, along with the morphisms fx_1 and fy_1, forms a cone over qk_2 and pk_2.

Due to the defining property of the pullback, there must be a unique morphism $Ff : F(R_1, k_1) \to F(R_2, k_2)$ such that the following diagram commutes:



If we apply the same process to a morphism $g : (R_2, k_2) \to (R_3, k_3)$, where (R_3, k_3) is any object, we will find that Fg is the unique morphism such that $x_3Fg = gx_2$ and $y_3Fg = gy_2$. In addition, $F(gf)$ is the unique morphism such that $x_3F(gf) = gfx_1$ and $y_3F(gf) = gfy_1$.

Note that $x_3FgFf = gx_2Ff = gfx_1$, and similarly, $y_3FgFf = gy_2Ff = gfy_1$. But I already said that $F(gf)$ is the *unique* morphism with those properties. Therefore, $FgFf$ must equal $F(gf)$, proving that F is a functor.

Define $s : F(A^2, id_{A^2}) \to A^2$ as the morphism $\langle px, qy \rangle$, where x and y are pullback projections and $qx = py$. Then a relation is transitive if and only if it preserves s.

To see how this is equivalent to the definition of transitivity stated earlier, assume there is some object (R, k) in C/A^2 with projections $x', y' : F(R, k) \to R$. Then Fk is the unique morphism such that $xFk = kx'$ and $yFk = ky'$. If (R, k) preserves s, then there exists $t : F(R, k) \to R$ satisfying $kt = sFk$. But $s = \langle px, qy \rangle$, so $kt = \langle pxFk, qyFk \rangle = \langle pkx', qky' \rangle$.

In summary, several properties of relations, including reflexivity, symmetry, and transitivity, can be stated in terms of "preserving" some morphism. As a consequence, if two relations are both reflexive, their intersection is also reflexive, and likewise for symmetry and transitivity.