A Structure Theorem for Semimodules over Rings

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November 28, 2023

Abstract

Modules over rings are some of the most well-studied structures in algebra. By slightly weakening the definition of a module—removing the requirement of an additive identity and additive inverses—one obtains the notion of a semimodule. The structure theorem for semimodules over rings, proven in this paper, states that there is a correspondence between semimodules over a commutative ring R and functors $F : X \to R\,\mathrm{Mod}$, where X is some semilattice. (I call such functors "nets of R-modules"; sheaves of R-modules on a topological space are a special case.) Specifically, there is an equivalence of categories between the category of R-semimodules and the category of nets of R-modules.

1 From semimodules to nets of modules

A **semimodule** over a commutative ring R is like a module over R, but with the additive structure of a commutative semigroup instead of an abelian group. Explicitly, a semimodule over R (or R-semimodule) is a set A equipped with binary operations $+ : A \times A \to A$ and $\cdot : R \times A \to A$ (with the latter also denoted by juxtaposition) such that:

- $+$ is commutative and associative,
- $1a = a$,
- $(rs)a = r(sa)$,
- $(r + s)a = ra + sa$,
- $r(a + b) = ra + rb$,

for all $r, s \in R$ and $a, b \in A$.

Proposition 1.1. *For any R-semimodule A and any element $a \in A$, the following are equivalent:*

1. $a + a = a$,
2. $0a = a$,

3. $ra = a$ *for all* $r \in R$,

4. $0x = a$ *for some* $x \in A$.

Proof. $3 \implies 2$ and $2 \implies 4$ are obvious.

$2 \implies 3$ because if $0a = a$, then $ra = r(0a) = (r0)a = 0a = a$.

$4 \implies 1$ because if $0x = a$, then $a + a = 0x + 0x = (0 + 0)x = 0x = a$.

$1 \implies 2$ because if $a+a = a$, then $0a = (1-1)a = a+(-1)a = a+a+(-1)a = (1 + 1 - 1)a = a$. $\square$

For A an R-semimodule, I will define $Z_A \subseteq A$ to be the subset consisting of elements of A which satisfy one of the four equivalent properties in proposition 1.1. I will define $0_A : A \to A$ to be the function $0_A(x) = 0x$. The following propositions are easily proven:

Proposition 1.2. *Z_A is a sub-semimodule of A.*

Proposition 1.3. *0_A is an idempotent R-linear map (i.e. a projection), and the image of 0_A is Z_A.*

Since every element $x \in Z_A$ is idempotent (in the sense that $x + x = x$), proposition 1.2 implies that Z_A is a semilattice in the algebraic sense, and thus also a semilattice in the order-theoretic sense. In other words, the relation $\leq$ defined by

$$a \leq b \iff \text{there is some } x \in Z_A \text{ such that } a + x = b$$

is a partial order on Z_A, and the least upper bound of any two elements with respect to this partial order is their sum.

Proposition 1.4. *For any R-semimodule A and any $a \in Z_A$, the fiber of 0_A at a, i.e. the subset $\{x \in A \mid 0x = a\}$, is an R-module with a as its additive identity.*

Proof. I will denote the subset in question as $M_A(a)$. It is easy to show that $M_A(a)$ is closed under addition and scalar multiplication, and is thus a sub-semimodule of A. Moreover, for any $x \in M_A(a)$, one has $a + x = x$, $(-1)x \in M_A(a)$, and $x + (-1)x = a$, so $M_A(a)$ is in fact an R-module. $\square$

Since the domain of any function is a disjoint union of its fibers, proposition 1.4 implies that every R-semimodule is a disjoint union of R-modules.

The next three lemmas show that any relationship $a \leq b$ with $a, b \in Z_A$ can be lifted to a map from $M_A(a)$ to $M_A(b)$ in a functorial way.

Proposition 1.5. *For any R-semimodule A, and any $a, b \in Z_A$ such that $a \leq b$, there is a well-defined R-linear map $M_A(a, b) : M_A(a) \to M_A(b)$ defined by $M_A(a, b)(x) = x + z$, where z is some element of Z_A such that $a + z = b$.*

Proof. If z and z' are elements of Z_A such that $a+z = a+z' = b$, and $x \in M_A(a)$, then $x + z = x + a + z = x + a + z' = x + z'$. Thus $M_A(a, b)$ is well-defined in the sense that it does not depend on the choice of z.

If $x \in M_A(a)$ and $z \in Z_A$ such that $a + z = b$, then $0(x + z) = 0x + 0z = a+z = b$, so $x+z \in M_A(b)$. Thus $M_A(a, b)$ does in fact map $M_A(a)$ into $M_A(b)$.

Lastly, $M_A(a, b)$ is an R-linear map:

$$\begin{gathered} M_A(a,b)(x+y) = x+y+z = x+y+z+z = \\ x+z+y+z = M_A(a,b)(x) + M_A(a,b)(y). \\ M_A(a,b)(rx) = rx+z = rx+rz = r(x+z) = rM_A(a,b)(x). \end{gathered}$$ $\square$

Proposition 1.6. *For any R-semimodule A and any $a \in Z_A$, the map $M_A(a, a) : M_A(a) \to M_A(a)$ is the identity.*

Proof. By definition, $M_A(a, a)(x) = x + z$ where z is an element of Z_A such that $a + z = a$. The choice of z does not matter, so we can take $z = a$. Then $M_A(a, a)(x) = x + a = x$, since $x \in M_A(a)$. $\square$

Proposition 1.7. *For any R-semimodule A and any $a, b, c \in Z_A$ such that $a \leq b \leq c$, $M_A(b, c) \circ M_A(a, b) = M_A(a, c)$.*

Proof. Let z, z' be elements of Z_A such that $a + z = b$ and $b + z' = c$. Note that $a+z+z' = c$. Therefore, $M_A(b, c)(M_A(a, b)(x)) = M_A(b, c)(x+z) = x+z+z'$, and $M_A(a, c)(x) = x + z + z'$. $\square$

I have constructed, for every R-semimodule A, a functor $M_A : Z_A \to R\,\mathrm{Mod}$, where Z_A is viewed as a thin category and $R\,\mathrm{Mod}$ is the category of R-modules and R-linear maps. This construction extends to a functor from the category of R-semimodules to a certain category of functors, as I will lay out in the next four lemmas.

Proposition 1.8. *For any R-linear map $f : A \to B$ between semimodules, and any $a \in Z_A$, $f(a) \in Z_B$. In other words, f restricts to a homomorphism from Z_A to Z_B.*

Proposition 1.9. *For any R-linear map $f : A \to B$ between semimodules, there is a natural transformation $f^* : M_A \to M_B \circ f$,*

$$\begin{tikzcd} Z_A \arrow[rr, "f"] \arrow[dr, "M_A"'] \arrow[rr, phantom, shift right=6ex, "\overset{f^*}{\Longrightarrow}"] & & Z_B \arrow[dl, "M_B"] \\ & R\,\mathrm{Mod} & \end{tikzcd}$$

such that the component of f^* at point $a \in Z_A$ is simply the restriction of f to $M_A(a)$.

Proof. For this to be well-defined, the restriction of f to $M_A(a)$ must produce a map into $M_B(f(a))$. Indeed, if x is such that $0x = a$, then $0f(x) = f(0x) = f(a)$.

The naturality condition for f^* states that, for every $a, a' \in Z_A$ such that $a \leq a'$, the following square commutes:

$$\begin{tikzcd} M_A(a) \arrow[r, "M_A(a,a')"] \arrow[d, "f"'] & M_A(a') \arrow[d, "f"] \\ M_B(f(a)) \arrow[r, "M_B(f(a),f(a'))"'] & M_B(f(a')). \end{tikzcd}$$

To this end, let z be some element of Z_A such that $a + z = a'$, and let x be any element of $M_A(a)$. Then

$$f(M_A(a,a')(x)) = f(x+z) = f(x) + f(z), \text{ and}$$
$$M_B(f(a), f(a'))(f(x)) = f(x) + f(z),$$

where the second equation is justified by the fact that $f(a) + f(z) = f(a')$. $\square$

Proposition 1.10. *Let A be any R-semimodule, and let i be the identity map on A. Then i^* is the identity natural transformation on M_A.*

Proposition 1.11. *Let A, B, and C be R-semimodules, and let $f : A \to B$ and $g : B \to C$ be R-linear maps. Then $(g \circ f)^* = g^* f \circ f^*$, where $g^* f : M_B \circ f \to M_C \circ g \circ f$ is the "whiskering" of g^* with f, i.e. the natural transformation whose component at $a \in Z_A$ is the restriction of g to $M_B(f(a))$.*

Proof. The component of $(g \circ f)^*$ at $a \in Z_A$ is the restriction of $g \circ f$ to $M_A(a)$, and the component of $g^* f \circ f^*$ at $a \in Z_A$ is composite of the restriction of g to $M_B(f(a))$ with the restriction of f to $M_A(a)$. These are clearly equal. $\square$

Given some category C, I will define $\mathrm{Net}(C)$ to be the category

- whose objects are pairs (X, F), where X is some semilattice and $F : X \to C$ is a functor,
- and where a morphism from (X, F) to (X', F') is a pair (g, τ), where $g : X \to X'$ is a map of semilattices and $\tau : F \to F' \circ g$ is a natural transformation.

Composition in $\mathrm{Net}(C)$ is defined in a straightforward way involving "whiskering." (I will omit the proof that this is in fact a category.) I will call the objects of $\mathrm{Net}(C)$ "nets" by analogy with another use of this term: in topology, a net is a map from a directed set into a space, and similarly, an object of $\mathrm{Net}(C)$ is a map from a semilattice (a specific type of directed set) into C.

In propositions 1.8 through 1.11, I have shown that there is a functor from $R\,\mathrm{Semimod}$ (the category of R-semimodules) to $\mathrm{Net}(R\,\mathrm{Mod})$, whose value at a semimodule A is the pair (Z_A, M_A), and whose value at a map of semimodules $f : A \to B$ is the pair (f, f^*). I will denote this functor as Struct : $R\,\mathrm{Semimod} \to \mathrm{Net}(R\,\mathrm{Mod})$, because it maps an R-semimodule to its "internal structure."

The *structure theorem for semimodules over rings* states that (for any commutative ring R) the functor Struct is an equivalence of categories, or in other words, R-semimodules are equivalent to nets of R-modules in a functorial way. In the following two sections, I will prove this theorem by explicitly constructing a functor from $\mathrm{Net}(R\,\mathrm{Mod})$ to $R\,\mathrm{Semimod}$ and showing that it is inverse to Struct.

2 From nets of modules to semimodules

Proposition 2.1. *For any semilattice X and any functor $F : X \to R\,\mathrm{Mod}$, the disjoint union*

$$\amalg F = \coprod_{x \in X} F(x)$$

is an R-semimodule, with scalar multiplication defined in the obvious way and addition defined as follows: for $a \in F(x)$ and $b \in F(y)$, $a + b = F(x, x+y)(a) + F(y, x+y)(b)$, where the addition on the right is in $F(x+y)$.

Proof. The identities $1a = a$, $(rs)a = r(sa)$ and $(r+s)a = ra+sa$ hold trivially. The addition operation on $\amalg F$ is commutative because addition in X and in $F(x+y)$ is commutative. The left distributive law holds because for any $r \in R$, $a \in F(x)$, and $b \in F(y)$,

$$r(a+b) = r(F(x, x+y)(a) + F(y, x+y)(b)) =$$
$$F(x, x+y)(ra) + F(y, x+y)(rb) = ra + rb.$$

Lastly, the proof of associativity is also rather routine, and makes use of the functoriality of F. □

Proposition 2.2. *For any morphism $(g, \tau) : (X, F) \to (X', F')$ in $\mathrm{Net}(R\,\mathrm{Mod})$, there is a map of semimodules $\tau_* : \amalg F \to \amalg F'$ defined by $\tau_*(a \in F(x)) = \tau_x(a)$.*

Proof. τ_* preserves scalar multiplication because, for any $r \in R$ and $a \in F(x) \subseteq \amalg F$, $\tau_*(ra) = \tau_x(ra) = r\tau_x(a) = r\tau_*(a)$.

τ_* preserves addition because, for any $a \in F(x)$ and $b \in F(y)$,

$$\begin{aligned}
\tau_*(a+b) &= \tau_*(F(x, x+y)(a) + F(y, x+y)(b)) \\
&= \tau_{x+y}(F(x, x+y)(a) + F(y, x+y)(b)) \\
&= \tau_{x+y}(F(x, x+y)(a)) + \tau_{x+y}(F(y, x+y)(b)) \\
&= F'(g(x), g(x+y))(\tau_x(a)) + F'(g(y), g(x+y))(\tau_y(b)) \\
&= F'(g(x), g(x)+g(y))(\tau_x(a)) + F'(g(y), g(x)+g(y))(\tau_y(b)) \\
&= \tau_x(a) + \tau_y(b) \\
&= \tau_*(a) + \tau_*(b).
\end{aligned}$$

□

Proposition 2.3. *Let X be a semilattice, $F : X \to R\,\mathrm{Mod}$ a functor, and $i : F \to F$ the identity natural transformation on F. Then $i_* : \amalg F \to \amalg F$ is the identity function on $\amalg F$.*

Proposition 2.4. *Let $(X, F) \xrightarrow{(g,\tau)} (X', F') \xrightarrow{(g',\tau')} (X'', F'')$ be a chain of morphisms in $\mathrm{Net}(R\,\mathrm{Mod})$. Then $(\tau' g \circ \tau)_* = \tau'_* \circ \tau_*$.*

Proof. Let a be an element of $\amalg F$ such that $a \in F(x)$, where $x \in X$. Then

$$(\tau' g \circ \tau)_*(a) = (\tau' g \circ \tau)_x(a) = \tau'_{g(x)}(\tau_x(a)) = \tau'_*(\tau_*(a)). \qquad \square$$

In propositions 2.1 through 2.4, I have constructed a functor $\mathrm{Total} : \mathrm{Net}(R\,\mathrm{Mod}) \to R\,\mathrm{Semimod}$ defined on objects as $\mathrm{Total}(X, F) = \amalg F$ and on morphisms as $\mathrm{Total}(g, \tau) = \tau_*$. In the next section, I will show that Struct and Total are inverses, and thus constitute an equivalence of categories between $R\,\mathrm{Semimod}$ and $\mathrm{Net}(R\,\mathrm{Mod})$.

3 Equivalence of categories

Proposition 3.1. *For any R-semimodule A,* $\mathrm{Total}(\mathrm{Struct}(A)) = A$.

Proof. Recall that $\mathrm{Struct}(A)$ is the object (Z_A, M_A) where Z_A and M_A are defined as in section 1. So $\mathrm{Total}(\mathrm{Struct}(A))$ is, as a set, the disjoint union

$$\coprod_{x \in Z_A} M_A(x).$$

As previously noted, the subsets of the form $M_A(x)$ are pairwise disjoint and their union is all of A, so the expression above is clearly naturally isomorphic, if not equal, to A.

The addition operation on $\mathrm{Total}(\mathrm{Struct}(A))$ is defined by

$$a + b = M_A(x, x + y)(a) + M_A(y, x + y)(b)$$

for $a \in M_A(x)$ and $b \in M_A(y)$. This is the same as the addition operation on A, because

$$M_A(x, x + y)(a) + M_A(y, x + y)(b) = a + y + b + x = a + x + b + y = a + b. \ \square$$

Proposition 3.2. *For any object $(X, F) \in \mathrm{Net}(R\,\mathrm{Mod})$,* $\mathrm{Struct}(\mathrm{Total}(X, F)) = (X, F)$.

Proof. By definition, $\mathrm{Struct}(\mathrm{Total}(X, F)) = (Z_{\amalg F}, M_{\amalg F})$. $Z_{\amalg F}$ is the subset $\{x \in \amalg F \mid 0x = 0\}$, which is simply the subset containing the additive identity of each module in the image of F. This set can be naturally identified with X by equating each $x \in X$ with the additive identity of $F(x)$.

Under this identification, $M_{\amalg F}$ is the functor taking $x \in X$ to the set of elements $a \in \amalg F$ such that $0a$ is the additive identity of $F(x)$. Clearly, this is the case if and only $a \in F(x)$. In other words, $M_{\amalg F}$ takes $x \in X$ to $F(x)$, so $M_{\amalg F} = F$. $\square$

Proposition 3.3. *For any morphism of R-semimodules $f : A \to B$,* $\mathrm{Total}(\mathrm{Struct}(f)) = f$*.*

Proof. $\mathrm{Struct}(f) = (f, f^*)$, so $\mathrm{Total}(\mathrm{Struct}(f)) = (f^*)_*$. By definition, for $a \in M_A(x)$, $(f^*)_*(a) = (f^*)_x(a) = f(a)$. $\square$

Proposition 3.4. *For any morphism $(g, \tau) : (X, F) \to (X', F')$ in* $\mathrm{Net}(R\,\mathrm{Mod})$, $\mathrm{Struct}(\mathrm{Total}(g, \tau)) = (g, \tau)$*.*

Proof. $\mathrm{Total}(g, \tau) = \tau_*$, so $\mathrm{Struct}(\mathrm{Total}(g, \tau)) = (\tau_*, (\tau_*)^*)$. τ_* maps the additive identity of $F(x)$ to the additive identity of $F'(g(x))$, so by the usual identification, τ_* is the same as g when evaluated on elements of X. And $(\tau_*)^*$ is the natural transformation whose component at $x \in X$ is the restriction of τ_* to $F(x) \subseteq \amalg F$. But on $F(x)$, τ_* is simply defined as τ_x, so $(\tau_*)^* = \tau$. $\square$